PART I (Calculators Not Allowed)

1. Calculate \( \int \left( \sin x + 2x + \frac{3}{x} \right) \, dx \).

(a) \(-\cos x + x + \ln |x| + C\)
(b) \(-\cos x + x^2 + 3 \ln |x| + C\)
(c) \(-\cos x + \frac{x^3}{3} + \frac{3}{x} + C\)
(d) \(-\sin x + x^2 + 3 \ln |x| + C\)
(e) \(\sin x + 2 - \frac{3}{x^2} + C\)

2. Calculate the definite integral \( \int_0^\pi x \cos(x^2) \, dx \).

(a) \(2(\sin \pi)^2\)
(b) \(\frac{\sin(\pi^3)}{3}\)
(c) \(\frac{\cos(\pi^2)}{2}\)
(d) \(\frac{\sin(\pi^2)}{2}\)
(e) \(\sin(\pi^2)\)

3. Which of the following definite integrals gives the length of the curve \( y = \ln x \) for \( 1 \leq x \leq e \).

(a) \( \int_1^e \sqrt{1 + \frac{1}{x^2}} \, dx \)
(b) \( \int_1^e \sqrt{1 + x^2} \frac{dx}{x^2} \)
(c) \( \int_1^e \sqrt{1 + x} \, dx \)
(d) \( \int_1^e \sqrt{1 + (\ln x)^2} \, dx \)
(e) \( \int_1^e \sqrt{1 + x^2} \, dx \)
4. \( \int_{3}^{6} (x - 2)e^x \, dx = \)

(a) \(4e^6 + e^3\)
(b) \(3e^6\)
(c) \(6e^6 - 3e^3\)
(d) \((x - 3)e^x + C\)
(e) \(e^{x-3}\)

5. \( \int \frac{dx}{(x + 3)^2 + 1} = \)

(a) \(\frac{1}{3} \tan^{-1}(x + 3) + C\)
(b) \(\sin^{-1}(x + 3) + C\)
(c) \(\tan^{-1}(x + 3) + C\)
(d) \(\ln[(x + 3)^2 + 1] + C\)
(e) \(e^{x+2}\)

6. Which of the following statements is correct?

(a) \(\sum_{n=1}^{\infty} \frac{1}{n^2}\) converges, and \(\sum_{n=1}^{\infty} \frac{\sin n}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}\).

(b) \(\sum_{n=1}^{\infty} \frac{1}{n^2}\) converges, but \(\sum_{n=1}^{\infty} \frac{\sin n}{n^2}\) diverges.

(c) \(\sum_{n=1}^{\infty} \frac{1}{n^2}\) converges, and \(\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}\)

(d) \(\sum_{n=1}^{\infty} \frac{1}{n^4}\) converges, and \(\sum_{n=1}^{\infty} \frac{\sin n}{n^4} > \sum_{n=1}^{\infty} \frac{1}{n^4}\)

(e) \(\sum_{n=1}^{\infty} \frac{1}{n^2}\) converges, but \(\sum_{n=1}^{\infty} \frac{1}{n^3}\) diverges.
7. Which of the following statements is correct?

(a) \( \int_0^\infty e^{-x} \, dx \) diverges
(b) \( \int_0^\infty e^{-x} \, dx \) converges and \( \int_0^\infty e^{-x} \, dx > e \).
(c) \( \int_0^\infty e^{-x} \, dx \) converges and \( \int_0^\infty e^{-x} \, dx = 1 \).
(d) \( \int_0^\infty e^{-x} \, dx \) converges and \( \int_0^\infty e^{-x} \, dx = 0 \).
(e) Both \( \int_0^\infty e^{-x} \, dx \) and \( \int_0^\infty e^x \, dx \) converge.

8. Choose the definite integral that is equal to the limit \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^3}{n^4} \).

(a) \( \int_0^1 x^4 \, dx \)
(b) \( \int_1^n \frac{1}{x} \, dx \)
(c) \( \int_0^1 \frac{1}{x} \, dx \)
(d) \( \int_0^1 x^3 \, dx \)
(e) \( \int_{1/n}^1 \frac{1}{x} \, dx \)

9. Since \( \int \frac{du}{u^2 \sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C \) and \( 9 - 8x - x^2 = 25 - (x + 4)^2 \),

one gets \( \int \frac{dx}{(x + 4)^2 \sqrt{9 - 8x - x^2}} = \)

(a) \( -\frac{\sqrt{9 - 8x - x^2}}{5(x + 4)} + C \)
(b) \( -\frac{\sqrt{25 - x^2}}{25x} + C \)
(c) \( -\frac{\sqrt{9 - 8x - x^2}}{25x + 100} + C \)
(d) \( -\frac{\sqrt{9 - 8x - x^2}}{25x^2} + C \)
(e) \( \frac{\sqrt{9 - 8x - x^2}}{25x + 4} + C \)
10. Suppose \( \int_0^2 f(x) \, dx = 5, \int_2^5 f(x) \, dx = 5, \int_1^4 f(x) \, dx = 1 \), then compute \( \int_1^5 f(x) \, dx \).

(a) \( \int_1^5 f(x) \, dx = 11 \).
(b) \( \int_1^5 f(x) \, dx = 9 \).
(c) \( \int_1^5 f(x) \, dx \) cannot be calculated from the given information.
(d) \( \int_1^5 f(x) \, dx = -1 \).
(e) \( \int_1^5 f(x) \, dx = -9 \).

11. The function \( f(x) = \int_0^x \sqrt{1+t^3} \, dt \) has the derivative

(a) \( f'(x) = 2x\sqrt{1+x^3} \)
(b) \( f'(x) = (2x-1)\sqrt{1+x^3} \)
(c) \( f'(x) = x^2\sqrt{1+x^3} \)
(d) \( f'(x) = \frac{x^3\sqrt{1+x^3}}{3} \)
(e) \( f'(x) = 2x\sqrt{1+x^3} \)

12. The MacLaurin series for the function \( f(x) = e^{2x} - 1 \) has the first four terms

(a) \( 2x + 2x^2 + \frac{4}{3} x^3 + \frac{2}{3} x^4 \)
(b) \( 1 + 2x + 2x^2 + \frac{2}{3} x^3 \)
(c) \( 2x + 2x^2 + \frac{2}{3} x^3 + 4x^4 \)
(d) \( 2x + x^2 + \frac{1}{3} x^3 + \frac{1}{12} x^4 \)
(e) \( 1 + 2x + x^2 + \frac{1}{3} x^3 \)

13. The sequence with the general term \( a_n = \frac{(-1)^n}{n^2} \)

(a) diverges to \(-\infty\).
(b) converges to \( \frac{\pi^2}{12} \).
(c) converges to 0.
(d) converges to \( n^2 - 1 \).
(e) none of the above.
14. A particle moves along a line to the right, starting at \( x = 0 \) at time \( t = 0 \). Its velocity at time \( t \) is \( v(t) = 2 + 3t \). Find the displacement of the particle at time \( t = 10 \).

(a) 32  
(b) \( +\infty \)  
(c) 170  
(d) 320  
(e) 190

15. The function \( f(x) = \cos(x^2) \) has the MacLaurin series

(a) \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n} \]  
(b) \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \]  
(c) \[ \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \]  
(d) \[ \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \]  
(e) \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \]

16. The series \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \) represents the function

(a) \( \cos x \)  
(b) \( e^x + e^{-x} \)  
(c) \( e^{-x^2} \)  
(d) \( \frac{1}{1 + x^2} \)  
(e) \( \frac{1}{x^2 - 1} \)
17. Use your calculator to get the average value of the function \( f(x) = (\sin x)^3 \) on the interval \( 0 \leq x \leq \pi \).

(a) \( \frac{1}{2} \)
(b) \( \frac{2}{\pi} \)
(c) \( \frac{4}{3\pi} \)
(d) \( \frac{4}{3} \)
(e) \( \frac{\pi}{2} \)

18. Which one of the of the following is true: The formula

\[
\frac{b-a}{3n} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n) \right]
\]

with \( n \) even and \( x_i = a + i \frac{b-a}{n} \), gives the approximation of the integral \( \int_a^b f(x)dx \) by the

(a) Trapezoid rule
(b) Midpoint rule
(c) Left hand sum
(d) Simpson rule
(e) Right hand sum

19. As a reminder: With \( x_i = a + i \frac{b-a}{n} \), the trapezoid rule is:

\[
T_n = \frac{b-a}{2n} \left[ f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \right]
\]

Compute the integral \( \int_0^1 e^{-x^2} \, dx \) using the trapezoid rule with \( n = 4 \).

(a) 0.6639690279
(b) 0.7429840978
(c) 0.8219991677
(d) 1.490678862
(e) 1.275893633
20. The volume of the body generated by rotating the curve $y = e^x$ with $0 \leq x \leq 1$ about the $x$-axis is given by

(a) $\pi \int_0^1 e^x \, dx$

(b) $\int_0^1 xe^x \, dx$

(c) $\frac{\pi(e^2 - 1)}{2}$

(d) $\pi \int_0^1 xe^x \, dx$

(e) $2\pi e$

21. The volume of the body generated by rotating the area under the curve $y = x^3$ with $0 \leq x \leq 5$ about the vertical, or $y$-axis is given by

(a) $\pi \frac{7^6}{7}$

(b) $\frac{5^7}{7}$

(c) $2\pi \frac{5^7}{7}$

(d) $1250\pi$

(e) $1250$

22. Determine the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x - 5)^n$

(a) $[4, 6]$

(b) $(4, 6]$

(c) $(4, 6)$

(d) $(\infty, 6)$

(e) $(4, \infty)$
23. One can check that $\frac{1}{6} < \sum_{n=6}^{\infty} \frac{1}{n^2} < \frac{1}{5}$. Sum up the first five missing terms and check which of the following holds, for the infinite sum $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$:

(a) $S = 1.463611111$
(b) $1.630277778 < S < 1.663611111$
(c) $1.630277778 > S > 1.25$
(d) $S = 1.663611111$
(e) $1.463611111 < S < 1.630277778$

24. Find the exact value of the geometric series $\sum_{n=0}^{\infty} (\pi - 3)^n$.

(a) $7.062513306$
(b) $\pi - 2$
(c) $\frac{\pi - 3}{4 - \pi}$
(d) $\frac{1}{4 - \pi}$
(e) $\infty$

25. Take the area bounded by the the $x$-axis, the parabola $f(x) = x^2$, and the vertical line $x = 2$. From the formula given, calculate the $y$-coordinate of the center of mass:

$$\bar{y} = \frac{\int_0^2 [f(x)]^2 \, dx}{2 \int_0^2 f(x) \, dx}$$

(a) $\bar{y} = 1$
(b) $\bar{y} = \frac{6}{5}$
(c) $\bar{y} = \frac{3}{10}$
(d) $\bar{y} = \frac{3}{4}$
(e) $\bar{y} = \frac{4\pi}{9}$
1. (a) (4 points) Find the area of the bounded region lying between the curves $y = x^3$ and $y = x^2$.

(b) (6 points) Set up and evaluate the integral for the volume generated by revolving the area from item (a) about the $x$-axis.
2. (a) (5 points) Given is an increasing function $y = f(x)$ with $f(0) = 0$ and $f(a) = b$. Consider the area with boundary consisting of the curve $y = f(x)$, the $x$-axis and the vertical line $x = a$. Set up, by the shell method, the integral for the volume $V$ generated by revolving that area about the vertical $y$-axis.

(b) (5 points) As an example for item (a), evaluate the integral for the volume generated by revolving the area between the curve $y = x^2$, the $x$-axis and the vertical line $x = 2$ about the $y$-axis.
3. A parabolically shaped trough is filled with water to a height of \( B = 2 \) m. We put the origin of the coordinate system at its deepest point. The boundary of the tank is obtained by rotating the curve \( y = x^2 \) about the vertical \( y \)-axis.

We have to find the work \( W \) gained by letting the water leak out at the bottom.

From physics, one needs that the acceleration due to gravity is \( g = 9.8 \) m/sec\(^2\), and the density of water \( \rho = 1000 \) kg/m\(^3\), hence \( \rho g = 9800 \) in these units. As a reminder: with \( y \) as variable of integration, the work is given by the integral

\[
W = \rho g \int_0^B y \pi x^2 \, dy
\]

(a) (5 points) Evaluate that work needed to pump the water.

(b) (5 points) Evaluate the work that is needed to pump all the water out to a height of \( H = 4 \) m above the deepest point of the trough.
4. (a) (5 points) Find \( \int \frac{x - 2}{\sqrt{x^2 - 4x + 5}} \, dx \), just using the substitution \( v = (x - 2)^2 \).

(b) (5 points) Find \( \int \frac{1}{\sqrt{x^2 - 4x + 5}} \, dx \). With another appropriate substitution, use the formula

\[
\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2}) + C
\]
5. (a) (4 points) To get the radius of convergence of the series \( \sum_{n=1}^{\infty} \frac{(3x)^n}{n(n+1)} \), calculate the ratio of consecutive terms and use the ratio test.

(b) (3 points) Use the formula \( \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \), write down a few terms.

Then evaluate the sum \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \).

(c) (3 points) What is the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(3x)^n}{n(n+1)} \)?

Is the series convergent at the endpoints of that interval? Use part (a) and (b) to get the answer.
6. We want to find the MacLaurin series of $\tan^{-1} x$. For simplicity, just give the first four terms of each of the following series. We begin with the geometric series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots$$

(a) (3 points) Use simple substitutions, and get the series representations for

$$\frac{1}{1 + x} =$$

$$\frac{1}{1 + x^2} =$$

(b) (3 points) Use part (a) to find a power series representation for

$$\tan^{-1} x = \int_0^x \frac{1}{1 + x^2} \, dx =$$

(c) (2 points) Calculate $\tan^{-1} 1$ in terms of $\pi$.

(d) (2 points) With $x = 1$ in part (b) you get the well-known series: $\frac{\pi}{4} =$