The Path-Connectivity of s-Elementary Frame Wavelets with Frame MRA

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Abstract. An s-elementary normalized tight frame wavelet (associated with an expansive matrix A as its dilation matrix) is a normalized tight frame wavelet whose Fourier transform is of the form $\frac{1}{\sqrt{2\pi}}\chi_E$ for some measurable set $E \subset \mathbb{R}^d$. It is known that the set of all such functions is path-connected. In this paper, we show that for any given $d \times d$ expansive matrix A, the set of all (A-dilation) s-elementary normalized tight frame wavelets with a frame MRA structure is also path-connected.

1. Introduction

A sequence $\{x_n\}$ in a Hilbert space H is called a *frame* for H if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq C_2\|x\|^2, \forall x \in H.$$

If $C_1 = C_2 = C$, $\{x_n\}$ is called a *tight frame* and the constant C is called the frame bound for $\{x_n\}$. In particular, if $C_1 = C_2 = 1$, then $\{x_n\}$ is called a *normalized tight frame*. It is known ([**9**]) that x_n is a normalized tight frame for H if and only if $x = \sum_{n\in\mathbb{N}}\langle x, x_n\rangle x_n$ for all $x \in H$. In this paper, we will use $L^2(\mathbb{R}^d)(= L^2(\mathbb{R}^d, \mu))$ as H, where $d \geq 1$ and μ is the Lebesgue measure. The set of all bounded linear operator acting on H is $B(H)$. A $d \times d$ matrix A is called an *expansive matrix* if all eigenvalues of A have modulus greater than one. Throughout this paper, A is understood to be an expansive matrix with integer entries (so that $A\mathbb{Z}^d \subset Z^d$).

Let T, D be the translation and dilation unitary operators acting on H defined by $(T^\ell f)(\mathbf{t}) = f(\mathbf{t} - \ell)$, $(D_A f)(\mathbf{t}) = |\det A|^{\frac{1}{2}} f(A\mathbf{t})$, $\forall f \in L^2(\mathbb{R}^d)$, $\mathbf{t} \in \mathbb{R}^d$. A function $\psi \in L^2(\mathbb{R}^d)$ is called a *frame wavelet* (*tight frame wavelet, normalized tight frame wavelet, orthonormal wavelet*) if $\{D_A^n T^\ell \psi : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$ is a frame (tight frame, normalized tight frame, orthonormal basis) for $L^2(\mathbb{R}^d)$.

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The topological property of various families of wavelets is an interesting topic in the study of wavelet theory. The question concerning the path-connectedness of the set of all orthonormal wavelets was first raised in [**6**]. Similar questions were raised and studied in [**2, 5, 10, 11, 12, 13**] concerning the sets of all MRA-wavelets, tight frame wavelets, MRA tight frame wavelets and a special class of frame wavelets called s-elementary frame wavelets (to be defined in the next section). In [**10, 13**], it is shown that the set of all MRA-wavelets is path-connected. In [**12**], it is shown that the set of all s-elementary orthonormal wavelets is path-connected. This result is extended to the set all s-elementary tight frame wavelets (with any given frame bound) in [**2**]. The proofs of these theorems were based on the complete characterizations of the corresponding wavelets. Interestingly, while the complete characterization of the s-elementary frame wavelets is still an open question, it has been shown that the set of s-elementary frame wavelets is path-connected as well [**5**]. In this paper, we will prove the path-connectedness of the s-elementary normalized tight frame wavelets with an additional structure called the *frame multiresolution analysis* (FMRA for short).

The basic definitions and preliminary results are given in the next section and the main theorem is stated and proved in Section 3.

2. s-elementary normalized tight frame wavelets with FMRA

The following definition is a natural generalization of standard multiresolution analysis (MRA) and is called the frame multiresolution analysis (FMRA). This was first introduced by Benedetto and Li [**1**].

Definition 2.1. A frame multiresolution analysis associated with a dilation matrix A (A-dilation FMRA for short) is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of H satisfying following conditions:

(1) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;

(2) $\bigcap_{j\in Z} V_j = \{0\}, \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;

(3) $D^j V_0 = V_j, DV_j = V_{j+1}, j \in \mathbb{Z}$;

(4) $W_0 = V_1 \ominus V_0, W_n = V_{n+1} \ominus V_n, DW_n = W_{n+1}$;

(5) There exists a function $\varphi \in V_0$ such that $\{T_\ell\varphi = \varphi(x-\ell), \ell \in \mathbb{Z}^d, x \in \mathbb{R}^d\}$ is a normalized tight frame for V_0.

The function φ in (5) above is called a frame scaling function for the A-dilation FMRA. A function $\psi \in W_0 = V_1 \ominus V_0$ is called an A-dilation normalized tight frame wavelet with FMRA if $\{T_\ell\psi(\mathbf{x}) = \psi(\mathbf{x}-\ell), \ell \in \mathbb{Z}^d, \mathbf{x} \in \mathbb{R}^d\}$ is a normalized tight frame for W_0. For the sake of simplicity, in the rest of this paper, such a function will be called an FMRA frame wavelet. Even though we do not mention the matrix A in this way, it is understood that all the FMRA frame wavelets are normalized tight frame wavelets with the same dilation matrix A. Furthermore, let us remind our reader that A has integer entries so we must have $|\det A| \geq 2$ since A is also expansive. In the above definition, if we replace "normalized tight frame" by "orthonormal basis " in (5), then we obtained the standard definition for MRA.

Let $f \in L^2(\mathbb{R})^d \cup L^1(\mathbb{R}^d)$, its Fourier-Plancherel transform is defined by

$$\hat{f}(\mathbf{s}) = (\mathcal{F}f)(\mathbf{s}) = \frac{1}{(2\pi)^{d/2}} \int_{R^d} e^{-i(\mathbf{s}\circ\mathbf{t})} f(\mathbf{t}) d\mathbf{t},$$

where $\mathbf{s} \circ \mathbf{t}$ denotes the real inner product. For a bounded linear operator S on $L^2(\mathbb{R})^d$, we will denote $\mathcal{F}S\mathcal{F}^{-1}$ by $\hat{S}$. It is easy to verify that $\hat{D_A} = D_{(A^\tau)^{-1}} = D^{-1}_{A^\tau} = D^*_{A^\tau}$ (A^τ is the transpose of A), and $\hat{T}_\lambda f = e^{-i(\lambda\circ\mathbf{s})} \cdot f$ for any $\lambda \in \mathbb{R}^d$.

Definition 2.2. Let E be a measurable set in $\mathbb{R}^d$. If $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E) = \psi_E$ is an FMRA frame wavelet, then ψ_E is called an s-elementary FMRA frame wavelet, and E is called an FMRA frame wavelet set.

In the case that E is an FMRA frame wavelet set, the function φ defined by $\hat{\varphi} = \frac{1}{\sqrt{\mu(K)}}\chi_K$, where $K = \bigcup_{m=1}^{\infty}(A^\tau)^{-m}E$ is a scaling function for the corresponding FMRA. It is easy to see that $K \subset A^\tau K$ and $E = A^\tau K \backslash K$. Let us call K the *scaling set* for E.

Two measurable sets E and F of $\mathbb{R}^d$ are 2π-translation congruent if there exists a measurable bijection $\theta: E \to F$ such that $\theta(\mathbf{t}) - \mathbf{t} \in 2\pi\mathbb{Z}^d$ for each $\mathbf{t} \in E$. Analogously, two measurable sets G and H are A-dilation congruent if there exists a measurable bijection $\xi: G \to H$ such that for any $\mathbf{t} \in G$, there exists $m \in \mathbb{Z}$ such that $\xi(\mathbf{t}) = A^m\mathbf{t}$. A measurable set E is a 2π-translation generator of $\mathbb{R}^d$ if $\{E + 2\ell\pi: \ell \in \mathbb{Z}^d\}$ forms a partition of $\mathbb{R}^d$. Analogously, a measurable set E is an A-dilation generator of $\mathbb{R}^d \setminus \{0\}$ if $\{A^m E: m \in \mathbb{Z}\}$ forms a partition of $\mathbb{R}^d \setminus \{0\}$.

Let us now list a few known results that we will need later in our proofs. The following lemma can be obtained using the same approach used in [**9**] for the one dimensional case.

Lemma 2.3. The following two statements hold:

(1) A measurable subset E of $\mathbb{R}^d$ is a normalized tight frame wavelet set if and only if E is both an A^τ-dilation generator of $\mathbb{R}^d$ and translation congruent to a subset of $[-\pi, \pi)^d$.

(2) Let K be a measurable set in $\mathbb{R}^d$ and $\varphi(\mathbf{t}) = \frac{1}{\sqrt{\mu(F)}}\chi_K(\mathbf{t})$, then $\{e^{i(\ell\circ\mathbf{t})}\varphi(\mathbf{t}), \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2(K)$ if and only if and K is translation congruent to a subset of $[-\pi, \pi)^d$.

Lemma 2.4. A measurable set E in $\mathbb{R}^d$ is an A-dilation FMRA frame wavelet set if and only if (1) E is an A^τ-dilation generator of $\mathbb{R}^d$ and is translation congruent to a subset of $[-\pi, \pi)^d$ and (2) $E = A^\tau K \backslash K$ for some K with the property that $K \subset A^\tau K$, and K is translation congruent to a subset of $[-\pi, \pi)^d$.

Proof. $\Longrightarrow$: By Lemma 2.3, (1) holds trivially. Let $\hat{\psi} = \hat{\psi_E} = \frac{1}{\sqrt{\mu(E)}}\chi_E$, and $K = \bigcup_{m=1}^{\infty}(A^\tau)^{-m}E$, $\hat{\varphi} = \frac{1}{\sqrt{\mu(K)}}\chi_K$. By the assumption, $\hat{T}^\ell\hat{\varphi} = \{e^{i\ell\circ x}\hat{\varphi}(\mathbf{x}), \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2(K)$. By Lemma 2.3, K is translation congruent to a subset of $[-\pi, \pi)^d$, and $A^\tau K = E \cup K$.

$\Longleftarrow$: By (1), E is a frame wavelet set. We will prove that $\psi = \mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$ is an A-dilation frame wavelet with FMRA. Let $W_j = \overline{\text{span}}\{D^j T^\ell \psi(\mathbf{x}), \ell \in \mathbb{Z}^d\}$, $j \in$

$\mathbb{Z}$, and $V_j = \bigoplus_{i=-\infty}^{j-1} W_i$. Then $\mathcal{F}(W_j) = L^2((A^\tau)^j E), j \in \mathbb{Z}$. Hence $\mathcal{F}(V_0) = L^2(\bigcup_{j=1}^{\infty}(A^\tau)^{-j}E)$. We have $K = \bigcup_{j=1}^{\infty}(A^\tau)^{-j}E$ by $K \subset A^\tau K$ and $E = A^\tau K \setminus K$. Since K is translation congruent to a subset of $[-\pi,\pi)^d$, it follows that $\hat{T}^\ell \frac{1}{\sqrt{\mu(K)}}\chi_K$ is a normalized tight frame for $L^2(K)$. Thus $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(K)}}\chi_K)$ is a scaling function in V_0. So $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$ is an A-dilation frame wavelet set with FMRA. □

LEMMA 2.5. [8] *For any real $d \times d$ expansive matrix A, there exists an open and bounded neighborhood F of the origin such that $F \subseteq A^\tau F$.*

LEMMA 2.6. [3] *Let A be a real expansive matrix. Then $lim_{k\to\infty}\|A^{-k}\| = 0$ and $lim_{k\to\infty}\|A^k\| = \infty$.*

THEOREM 2.7. *For every expansive matrix A, there exists an A-dilation FMRA frame wavelet set.*

PROOF. By Lemma 2.5, there exists an open and bounded neighborhood F of zero such that $F \subseteq A^\tau F$. Since F is bounded and $lim_{k\to\infty}\|(A^\tau)^{-k}\| = 0$ by Lemma 2.6, there is an integer k_0 such that $(A^\tau)^{k_0}F \subset B(1)$, where $B(1)$ is the unit ball with its center at the origin. Since $F \subset A^\tau F$, $(A^\tau)^{k_0-1}F \subset (A^\tau)(A^\tau)^{k_0-1}F$. Let $K = (A^\tau)^{k_0-1}F$, then $K \subset A^\tau K \subset B(1) \subset [-\pi,\pi)^d$. Let $E = A^\tau K \setminus K$. By the definition, $A^\tau K \setminus K$ and K are both translation congruent to some subsets of $[-\pi,\pi)^d$. Since K contains the origin as an interior point and A is expansive, we have that $\bigcup_{n\in\mathbb{Z}}(A^\tau)^n(A^\tau K\setminus K) = \mathbb{R}^d\setminus\{0\}$. Furthermore, $\{(A^\tau)^n(A^\tau K \setminus K) : n \in \mathbb{Z}\}$ are disjoint sets, hence $A^\tau K \setminus K$ is an A^τ-dilation generator of $\mathbb{R}^d$. The result now follows from Lemma 2.4. □

3. Path-connectivity of s-elementary FMRA frame wavelets

A set $S \subset L^2(\mathbb{R}^d)$ is said to be path-connected under norm topology of $L^2(\mathbb{R}^d)$ if for any two members f, $g \in S$, there exists a mapping $\gamma : [0,1] \to S$ such that the function $\gamma(t)$ is continuous in the norm of $L^2(\mathbb{R}^d)$ and $\gamma(0) = f$, $\gamma(1) = g$. However, in the case of wavelets or frame wavelets with either an MRA or FMRA structure, we would like the connecting path to preserve the corresponding scaling functions. In this paper, we consider the special case when S is the set of all s-elementary FMRA frame wavelets. The path that connects two such wavelets will need to satisfy one additional condition: the corresponding scaling functions of the points on the path (which are s-elementary FMRA frame wavelets) must form a continuous path connecting the scaling functions of two starting s-elementary FMRA frame wavelets. Let us call such a path a *scaling function preserving path*, or just an SP-path for short. Let E and F be two FMRA frame wavelet sets in $\mathbb{R}^d$ with K and N being their corresponding scaling sets. We will also say $\{E_t\}$ is a *scaling preserving path* (SP-path for short) connecting E and F if χ_{E_t} is a continuous (in the $L^2(\mathbb{R}^d)$ norm) path connecting χ_E and χ_F such that E_t is an FMRA frame wavelet set for each t and the corresponding scaling set K_t is also a continuous path connecting K and N. The following lemma is direct from this definition.

LEMMA 3.1. Let ψ_E and ψ_F be two s-elementary FMRA frame wavelets with φ_K and φ_N being their corresponding scaling function, E and F being their corresponding FMRA frame wavelet sets and K, N being the corresponding scaling sets of E, F. Then the following two statements are equivalent:

(1) ψ_E and ψ_F are connected by an SP-path.

(2) For each $t \in [0,1]$, there exist measurable sets E_t and K_t such that (a) $\{E_t\}$ is a normalized tight frame wavelet set with K_t being its scaling function set for each $t \in [0,1]$; (b) $E_0 = E$ and $E_1 = F$, $K_0 = K$ and $K_1 = N$; (c) E_t and K_t are continuous in t. That is, for any $t_0 \in [0,1] \cap (t_0 - \epsilon_1, t_0 + \varepsilon_1)$, we have $\mu(E_t \backslash E_{t_0}) + \mu(E_{t_0} \backslash E_t) < \varepsilon$ and $\mu(K_t \backslash K_{t_0}) + \mu(K_{t_0} \backslash K_t) < \varepsilon$.

THEOREM 3.2. *For any given $d \times d$ real expansive matrix A, the set of all A-dilation s-elementary FMRA frame wavelets is path-connected in the scaling function preserving sense.*

PROOF. Let ψ_E and ψ_F be two s-elementary FMRA frame wavelets with φ_K and φ_N being their corresponding scaling function, E and F being their corresponding FMRA frame wavelet sets and K, N being the corresponding scaling sets of E, F. The main idea of the proof is that we will find two special FMRA frame wavelet sets D and G and show that ψ_E and ψ_D are SP-path connected, ψ_F and ψ_G are SP-path connected, and ψ_D, ψ_G are also SP-path connected.

For the sake of convenience, let us introduce an operation notation: For any set $P \subset \mathbb{R}^d$, define $\Delta(P) = \cup_{j\in\mathbb{Z}}(A^\tau)^j P$ and $\Delta^-(P) = \cup_{j\geq 1}(A^\tau)^{-j} P$. By this definition, the scaling set of an FMRA frame wavelet set P is simply $\Delta^-(P)$.

By Theorem 2.7 (and the lemmas preceding it), there exists an FMRA frame wavelet set Q such that Q and $\Delta^-(Q)$ are both subsets of $[-\pi,\pi)^d$, and $\Delta^-(Q)$ is an open set that contains the origin. We have $K = \Delta^-(E)$. Consider the set $\Delta^-(Q) \cap \Delta^-(E)$.

CLAIM 3.3. *We claim that $D = A^\tau(\Delta^-(Q) \cap \Delta^-(E)) \setminus (\Delta^-(Q) \cap \Delta^-(E))$ is an FMRA frame wavelet set with $\Delta^-(D) = \Delta^-(Q) \cap \Delta^-(E)$ being its corresponding scaling set.*

Proof. By Lemma 2.4, to prove this claim, we need to establish the following: (1) $\Delta^-(D) \subseteq A^\tau \Delta^-(D)$; (2) $D = A^\tau \Delta^-(D) \setminus \Delta^-(D)$ and $\Delta^-(D)$ are both 2π-translation congruent to subsets of $[-\pi,\pi)^d$; (3) D is an A-dilation generator of $\mathbb{R}^d \setminus \{\mathbf{0}\}$.

First, we have $\Delta^-(D) = \Delta^-(Q) \cap \Delta^-(E) \subset A^\tau \Delta^-(Q) \cap A^\tau \Delta^-(E) = A^\tau(\Delta^-(Q) \cap \Delta^-(E))$, since $\Delta^-(Q) \subset A^\tau \Delta^-(Q)$ and $\Delta^-(E) \subset A^\tau \Delta^-(E)$. This proves (1). Secondly, since $\Delta^-(D) = \Delta^-(Q) \cap \Delta^-(E) \subseteq \Delta^-(Q)$ and $\Delta^-(Q)$ is a subset of $[-\pi,\pi)^d$, so is $\Delta^-(D)$. Furthermore, $D = A^\tau \Delta^-(D) \setminus \Delta^-(D) \subseteq A^\tau \Delta^-(Q) = Q \cup \Delta^-(Q)$. Since $Q \cup \Delta^-(Q)$ is a disjoint union and is a subset of $[-\pi,\pi)^d$, D and $\Delta^-(D)$ are disjoint subsets of $[-\pi,\pi)^d$. This proves (2). Thirdly, for any given $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $\{(A^\tau)^j \mathbf{x} : \ j \in \mathbb{Z}\}$ is a sequence of points that converges to the origin as $j \to -\infty$ by Lemma 2.6. Since E is an A-dilation generator, there exists $j_0 \in \mathbb{Z}$ such that $(A^\tau)^j \mathbf{x} \in \Delta^-(E)$ for any $j \leq j_0$. This implies that $\{(A^\tau)^j \mathbf{x} : \ j \in \mathbb{Z}\} \cap \Delta^-(D) \neq \emptyset$ since $\Delta^-(Q)$ is an open set containing O. On the other hand, since $A^\tau \mathbf{x}$ approaches infinity as $j \to \infty$ (again by Lemma 2.6), $(A^\tau)^j \mathbf{x} \notin \Delta^-(D)$ when j is large enough.

Thus, there exists $j_1 \in \mathbb{Z}$ such that $(A^\tau)^{j_1}\mathbf{x} \in \Delta^-(D)$ but $(A^\tau)^{j_1+1}\mathbf{x} \notin \Delta^-(D)$. This means $y = (A^\tau)^{j_1+1}\mathbf{x}$ is an element of $D = A^\tau\Delta^-(D) \setminus \Delta^-(D)$. So $\mathbf{x}$ is A-dilation equivalent to a point in D. Since $\mathbf{x}$ is arbitrary, this proves (3) and hence the claim.

Having established that the set D so defined above is an FMRA frame wavelet set with $\Delta^-(D) = \Delta^-(Q) \cap \Delta^-(E)$ being its corresponding scaling set, we are now in position to construct an SP-path connecting D and E. The approach used here is similar to the one used in [**2**] but we will provide the details here for the convenience of our reader.

Note 3.4. *In fact, the set D can be chosen to be very close to the origin such that for any point $\mathbf{x} \in D$, $A^\tau\mathbf{x} \in [-\pi,\pi)^d$. We will now assume that D is so chosen.*

Claim 3.5. *For any subset I of D, define $J = E \cap \Delta(I)$. We claim that*

(i) the set $E_I = (E \setminus J) \cup I$ is an A-dilation generator $\mathbb{R}^d \setminus \{\mathbf{0}\}$;

(ii) E_I and $\Delta^-(E_I)$ are disjoint and

(iii) $\Delta^-(E_I)$ is congruent to a subset of $[-\pi,\pi)^d$.

Proof. (i) is straight forward and is left to our reader to verify. (iii) holds since $\Delta^-(I)$ is actually a subset of $\Delta^-(E)$: $\Delta^-(E_I) = \Delta^-(E \setminus J) \cup \Delta^-(I) \subseteq \Delta^-(E)$ and $\Delta^-(E)$ is known to be congruent to a subset of $[-\pi,\pi)^d$. For (ii), again use $\Delta^-(E_I) = \Delta^-(E \setminus J) \cup \Delta^-(I)$. $\Delta^-(I)$ is disjoint from I since $I \subset D$ and D is disjoint from $\Delta^-(D)$. $\Delta^-(I)$ is also disjoint from $E \setminus J$ since it is actually contained in $\Delta^-(E)$. It is obvious that $\Delta^-(E \setminus J)$ is disjoint from $E \setminus J$. Finally, $\Delta(I) = \Delta(J)$ by the definition of J. It follows that I is disjoint from $\Delta(E \setminus J)$, but $\Delta^-(E \setminus J) \subset \Delta(E \setminus J)$. This concludes (ii).

For each $t \in [0,1]$, define $I_t^1 = [-\pi t, \pi t)^d \cap D$, $J_t^1 = E \cap \Delta(I_t^1)$ and $E_t^1 = (E \setminus J_t^1) \cup I_t^1$. E_t^1 is so defined as we wish to replace a part of E (namely J_t^1) by a part of D (namely I_t^1) that are A-dilation congruent (in a continuous manner) so that the resulting set $E_t^1 = (E \setminus J_t^1) \cup I_t$ is still an FMRA frame wavelet set. By Claim 3.5, $E_t^1 = (E \setminus J_t^1) \cup I_t^1$ is almost an FMRA frame wavelet set. The only extra condition that it needs to satisfy is that it must also be congruent to a subset of $[-\pi,\pi)^d$. The construction of E_t^1 apparently does not guarantee this. The following steps reflect our effort to modify E_t^1 so that this extra condition will hold at the end.

For any measurable set P, define $\mathcal{T}(P) = \cup_{\mathbf{0}\neq\ell\in\mathbb{Z}^d}(P + 2\pi\ell)$. Define $H_t^1 = (E \setminus J_t^1) \cap \mathcal{T}(I_t^1)$. In other word, H_t^1 is the part of $(E \setminus J_t^1)$ that would overlap with I_t^1 under non-trivial 2π-translations. We wish to get rid of it since our resulting set must be congruent to a subset of $[-\pi,\pi)^d$. Deleting H_t^1, of course, will then result in a deficiency in the set as an A-dilation generator of $\mathbb{R}^d \setminus \{\mathbf{0}\}$. So we will need to find a subset of D which will make up this deficiency. This sets up the following recursively defined sets.

$$
\begin{aligned}
I_t^2 &= (D \cap \Delta(H_t^1)) \cup I_t^1,\\
J_t^2 &= H_t^1 \cup J_t^1,\\
E_t^2 &= (E \setminus J_t^2) \cup I_t^2,\\
H_t^2 &= (E \setminus J_t^2) \cap \mathcal{T}(I_t^2),\\
I_t^3 &= (D \cap \Delta(H_t^2)) \cup I_t^2,\\
J_t^3 &= H_t^2 \cup J_t^2,\\
E_t^3 &= (E \setminus J_t^3) \cup I_t^3,\\
H_t^3 &= (E \setminus J_t^3) \cap \mathcal{T}(I_t^3),\\
\dots & \quad \dots
\end{aligned}
$$

By the above definition, $\{I_t^j : j \geq 1\}$ and $\{J_t^j : j \geq 1\}$ are ascending sequences. Now define $I_t = \cup_{j\geq 1} I_t^j$, $J_t = \cup_{j \geq 1} J_t^j$ and let $E_t = (E \setminus J_t) \cup I_t$. It is clear from the definition that $E_0 = E$ and $E_1 = D$. It is easy to see that $\Delta(I_t) = \Delta(J_t)$ hence E_t satisfies the conditions listed in Claim 3.5. If we can prove that E_t is congruent to a subset of $[-\pi, \pi)^d$, then E_t is an FMRA frame wavelet set for each t. Suppose this is not true. Then there exists $\mathbf{x} \in E \setminus J_t$ and a nontrivial $\ell \in \mathbb{Z}^d$ such that $\mathbf{x} + 2\pi\ell = \mathbf{s} \in I_t$. It follows that there exists $j_0 \in \mathbb{Z}$ such that $\mathbf{s} \in I_t^{j_0}$. Since $\mathbf{x} \in E \setminus J_t \subseteq E \setminus J_t^{j_0}$, we have $\mathbf{x} = \mathbf{s} - 2\pi\ell \in (E \setminus J_t^{j_0}) \cap \mathcal{T}(I_t^{j_0}) = H_t^{j_0} \subseteq J_t^{j_0+1} \subseteq J_t$, a contradiction.

What remains to be shown is that E_t is continuous in t.

For any $t_2 > t_1$, it is clear that $I_{t_1} \subseteq I_{t_2}$ and $J_{t_1} \subseteq J_{t_2}$ so $I_{t_1} \setminus I_{t_2} = \emptyset$ and $J_{t_1} \setminus J_{t_2} = \emptyset$. On the other hand, we have $I_{t_2} \setminus I_{t_1} = I'_{t_1,t_2}$, $J_{t_2} \setminus J_{t_1} = J'_{t_1,t_2}$ where I'_{t_1,t_2} and J'_{t_1,t_2} are defined the same way as I_t and J_t by replacing I_t^1 with $I_{t_1,t_2}^1 = ([-t_2\pi, t_2\pi)^d \setminus [-t_1\pi, t_1\pi)^d) \cap D$, whose measure is bounded above by $(2\pi t_2)^d - (2\pi t_1)^d$. First let us consider $I'_{t_1,t_2} = \cup_{j\geq 1} I_{t_1,t_2}^j$. We have $I_{t_1,t_2}^2 = (D \cap \Delta(H_{t_1,t_2}^1)) \cup I_{t_1,t_2}^1$. Since H_{t_1,t_2}^1 is 2π-translation congruent to a subset of I_{t_1,t_2}^1, its measure is at most the measure of I_{t_1,t_2}^1 (which is at most $(2\pi t_2)^d - (2\pi t_1)^d$). $D \cap \Delta(H_{t_1,t_2}^1)$ is the A-dilation equivalence of H_{t_1,t_2}^1 in D. By the definition of H_{t_1,t_2}^1, we have $H_{t_1,t_2}^1 \cap [-\pi, \pi)^d = \emptyset$. Thus by Note 3.4, each point of H_{t_1,t_2}^1 must be multiplied by a negative power of A^τ in order for the result to be in D. It follows that the measure of $D \cap \Delta(H_{t_1,t_2}^1)$ is at most $((2\pi t_2)^d - (2\pi t_1)^d)/|\det A|$. Following the same argument, we can show that $\mu(I_{t_1,t_2}^{j+1} \setminus I_{t_1,t_2}^j) \leq ((2\pi t_2)^d - (2\pi t_1)^d)/|\det A|^j$ in general. Since A is expansive, $|\det A| > 1$ hence the series $\sum_{j\geq 0} \frac{1}{|\det A|^j}$ converges to some constant b. It follows that

$$\mu(I'_{t_1,t_2}) \leq ((2\pi t_2)^d - (2\pi t_1)^d) \sum_{j\geq 0} \frac{1}{|\det A|^j} = b((2\pi t_2)^d - (2\pi t_1)^d).$$

This inequality then guarantees the continuity of I_t. The continuity of J_t can be similarly proved. Thus E_t is continuous (and so is ψ_{E_t}).

The SP-path between F and G (which is defined the same way as D with E replaced by F) can be similarly constructed.

Finally, we will construct the SP-path connecting D and G. Since D and G are both subsets of $[-\pi, \pi)^d$, this is much easier since there is no need to worry about the 2π-translation redundance so the path involves with only the first step as in the definition of E_t. $\square$

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