Multipliers, Phases and Connectivity of Wavelets in $L^2(\mathbb{R}^2)$

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Abstract. Let $A$ be any $2 \times 2$ real expansive matrix. For any $A$-dilation wavelet $\psi$, let $\hat{\psi}$ be its Fourier transform. A measurable function $f$ is called an $A$-dilation wavelet multiplier if the inverse Fourier transform of $(f \hat{\psi})$ is an $A$-dilation wavelet for any $A$-dilation wavelet $\psi$. In this paper, we give a complete characterization of all $A$-dilation wavelet multipliers under the condition that $A$ is a $2 \times 2$ matrix with integer entries and $|\det(A)| = 2$. Using this result, we are able to characterize the phases of $A$-dilation wavelets and prove that the set of all $A$-dilation MRA wavelets is path-connected under the $L^2(\mathbb{R}^2)$ norm topology for any such matrix $A$.

1. Introduction

Let $A$ be a $2 \times 2$ real expansive matrix, i.e., a matrix with real entries whose eigenvalues are all of modules greater than one. Let $L^2(\mathbb{R}^2)$ be the set of all square Lebesgue integrable functions in $\mathbb{R}^2$. An $A$-dilation wavelet is a function $\psi \in L^2(\mathbb{R}^2)$ such that the set

\[
\{|\det A|^{1/2} \psi(A^n t - \ell) : n \in \mathbb{Z}, \ell \in \mathbb{Z}^2\}
\]

forms an orthonormal basis for $L^2(\mathbb{R}^2)$. For any function $f(t) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, its Fourier transform is defined by

\[
\mathcal{F}(f(t)) = \hat{f}(s) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(t) e^{-it\cdot s} dt = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(t) e^{-it\cdot s} d\mu,
\]

where $\mu$ denotes the Lebesgue measure in $\mathbb{R}^2$ and $t \cdot s$ is the standard inner product of the vectors $s$, $t \in \mathbb{R}^2$. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$.

One of the many problems in wavelet theory concerns the construction of different wavelets. Naturally, one may attempt to construct new wavelets from an existing one. This approach leads to the concept of wavelet multipliers [4]. A measurable function $f$ is called an $A$-dilation wavelet multiplier if the inverse Fourier transform of $(f \hat{\psi})$ is an $A$-dilation wavelet for any $A$-dilation wavelet $\psi$. In the

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one dimensional case, wavelet multipliers have been studied extensively and completely characterized [8, 14, 17]. In the two dimensional case, a characterization of $A$-dilation wavelet multipliers is given in [13] for the following two $2 \times 2$ matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ or } A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$ Moreover, it is proven there that for any given $A$-dilation wavelet $\psi_0$ (under the above choice of $A$), the set $\mathcal{M}_{\psi_0} = \{ \psi : \hat{\psi} = v\hat{\psi}_0 \}$ where $v$ is an $A$-dilation wavelet multiplier is path-connected in $L^2(\mathbb{R}^2)$.

In this paper, we will generalize the above result to all $2 \times 2$ expansive matrix with integer entries such that $|\det(A)| = 2$. From this point on, we will always assume that $A$ is a $2 \times 2$ expansive matrix with integer entries such that $|\det(A)| = 2$. We will derive an explicit formula that can be used to construct all $A$-dilation wavelet multipliers for such matrices $A$. As applications of this result, we give a characterization of the phases of $A$-dilation MRA wavelets and prove that the set of all $A$-dilation MRA wavelets is path-connected in $L^2(\mathbb{R}^2)$.

The rest of the paper is organized as follows. In the next section, we introduce the notations and terms needed for this paper, with some preliminary results needed in the later sections. This is followed by Section 4 which includes several examples. In Section 5 we state and prove our main result on wavelet multipliers in the two dimensional case. In Section 6 we discuss the phases of $A$-dilation MRA wavelets and in Section 7 we prove the path-connectivity of the set of all $A$-dilation MRA wavelets. We end this paper with some open questions in Section 8.

2. Notations, Definitions and Preliminary Results

A matrix with integer entries will be called an integral matrix. As we mentioned in the introduction, our main interest in this paper concerns the case when $A$ is a $2 \times 2$ integral expansive matrix with $|\det(A)| = 2$. We will use $T$, $D_A$ as the translation and dilation unitary operators acting on $L^2(\mathbb{R}^2)$ defined by $(T^\ell f)(t) = f(t - \ell)$, $(D_A f)(t) = |\det(A)|^{\frac{\ell}{2}} f(At)$, $\forall f \in L^2(\mathbb{R}^2)$, $t \in \mathbb{R}^2$ and $\ell \in \mathbb{Z}^2$. A measurable function $\psi \in L^2(\mathbb{R}^2)$ is called an $A$-dilation wavelet if $\{D_A^n T^\ell \psi : n \in \mathbb{Z}, \ell \in \mathbb{Z}^2\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

**Definitions 2.1.** A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^2)$ is called an $A$-dilation multi-resolution analysis (or $A$-dilation MRA for short) if the following hold:

(i) $V_j \subset V_{j+1}$, $\forall j \in \mathbb{Z}$;
(ii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^2)$;
(iii) $f(t) \in V_j$ if and only if $f(A^{-j}t) \in V_0$ for $j \in \mathbb{Z}$; and
(iv) There exists $\phi(t)$ in $V_0$ such that $\{\phi(t - \ell) : \ell \in \mathbb{Z}^2\}$ is an orthonormal basis for $V_0$.

The function $\phi(t)$ defined in (iv) above is called an $A$-dilation scaling function for the MRA. In our case, it is known that a single $A$-dilation wavelet can be derived from the above $A$-dilation MRA [15] (due to the fact that $|\det(A)| - 1 = 1$). An
A-dilation wavelet \( \psi \) so obtained is called an MRA wavelet (and \( \psi \in V_1 \cap V_0^\perp \)). For any \( f \in V_1 \), \( f(A^{-1}t) \in V_0 \) hence we have

\[
f(t) = |\det(A)| \sum_{\ell \in \mathbb{Z}^2} c_\ell \phi(A \ell - t).
\]

If we define \( m_f(s) = \sum_{\ell \in \mathbb{Z}^2} c_\ell e^{-i \ell s} \), then by taking Fourier transform on both sides of (2.1) we obtain \( \hat{f}(A^T s) = m_f(s) \hat{\phi}(s) \), where \( A^T \) is the transpose of \( A \). In particular, we have

\[
\hat{\phi}(A^T s) = m(s) \hat{\phi}(s)
\]

for some function \( m(s) \) of the form similar to (2.1). \( m(s) \) is called the low pass A-dilation filter of the MRA.

A measurable function \( f \) is called an A-dilation wavelet multiplier if the inverse Fourier transform of \( \hat{f} \) is an A-dilation wavelet whenever \( \hat{\psi} \) is an A-dilation wavelet. A measurable function \( f(t) \in L^2(\mathbb{R}^2) \) is called a \( 2\pi \mathbb{Z}^2 \)-translation periodic if \( f(t + 2\pi \ell) = f(t) \) a.e. on \( \mathbb{R}^2 \) for any \( \ell \in \mathbb{Z}^2 \). \( f \) is called A-dilation periodic if \( f(A\ell) = f(t) \) a.e. on \( \mathbb{R}^2 \). Furthermore, \( f \) is called A-dilation-translation compatible if there exists a \( 2\pi \mathbb{Z}^2 \)-translation periodic function \( k(t) \) such that \( f(A \ell) = k(t)f(t) \). Apparently, the function \( m_f(s) \) and the low pass A-dilation filter defined above are \( 2\pi \mathbb{Z}^2 \)-translation periodic functions.

The following lemmas are well known results and can be easily obtained by standard textbook arguments [1, 6, 11]. Keep in mind that \( A \) stands for a \( 2 \times 2 \) integral expansive matrix with \( |\det A| = 2 \).

**Lemma 2.1.** \( \psi \) is an A-dilation wavelet iff the following conditions hold

(i) \( \| \psi \|_2 = 1 \);

(ii) \( \sum_{\ell \in \mathbb{Z}} |\hat{\psi}((A^T)^T s)|^2 = 1/(2\pi)^2 \) a.e. and

(iii) \( \sum_{j=0}^{\infty} \hat{\psi}((A^T)^T s)\overline{\hat{\psi}((A^T)^T(s + 2\pi \ell))} = 0 \) a.e. \( \forall \ell \in \mathbb{Z}^2 \setminus A^T \mathbb{Z}^2 \).

**Lemma 2.2.** An A-dilation wavelet \( \psi \) is an A-dilation MRA wavelet iff

\[
D_\psi(s) = \sum_{n=1}^{\infty} \sum_{\ell \in \mathbb{Z}^2} |\hat{\psi}((A^T)^n(s + 2\pi \ell))|^2 = \frac{1}{(2\pi)^2}, \text{ a.e.}
\]

**Lemma 2.3.** \( \phi \) is an A-dilation scaling function for an MRA iff the following conditions hold

(i) \( \sum_{\ell \in \mathbb{Z}^2} |\hat{\phi}(s + 2\pi \ell)|^2 = 1/(2\pi)^2 \) a.e.;

(ii) \( \lim_{j \to -\infty} |\hat{\phi}((A^T)^{-j} s)| = 1/2\pi \) a.e. and

(iii) there exists a \( 2\pi \mathbb{Z}^2 \)-translation periodic function \( m(s) \in L^2([-\pi, \pi]^2) \) such that \( \hat{\phi}(A^T s) = m(s)\hat{\phi}(s) \).
LEMMA 2.4. Suppose that $\psi$ is an $A$-dilation MRA wavelet with scaling function $\phi$, then

\[
|\hat{\phi}(s)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(A^j s)|^2, \text{ a.e.}
\]  

(2.4)

Since $|\det A| = 2$, the Abelian group $\mathbb{Z}^2/A^*\mathbb{Z}^2$ has only two elements [7]. We leave it to our reader to verify that there is a unique element $h_0 \in \mathbb{Z}^2 \setminus A^*\mathbb{Z}^2$ such that $(A^*)^{-1} h_0 = h_0$ is a non-zero vector whose entries are either 1/2 or 0. Let $u$ be a constant vector such that $h_0 \circ u = 1/2$. We have the following two propositions.

PROPOSITION 2.1. Let $\phi \in L^2(\mathbb{R}^2)$ be an $A$-dilation scaling function for an $A$-dilation MRA $\{V_j\}$ and let $m$ be its associated low pass filter. Let $\psi \in W_0 = V_0 \cap V_0^\perp$, then $\{\psi(t - \ell) : \ell \in \mathbb{Z}^2\}$ is an orthonormal basis for $W_0$ if

\[
\hat{\psi}(A^s t) = e^{i\phi_0 u} \hat{\phi}(s) m(s + 2\pi h_0) \hat{\phi}(s) \text{ a.e.}
\]  

(2.5)

where $v$ is a $2\pi\mathbb{Z}^2$-translation periodic measurable function with $|v(s)| = 1$ a.e. on $\mathbb{R}^2$.

Let us give an outline of the proof for Proposition 2.1. From the discussion following (2.1), we have $\hat{\psi}(A^s t) = m_\psi(s) \hat{\phi}(s)$ for some $2\pi\mathbb{Z}^2$-translation periodic function $m_\psi$. Again, standard arguments show that $\{\psi(t - \ell) : \ell \in \mathbb{Z}^2\}$ is an orthonormal basis for $W_0$ iff the equations $|m(s)|^2 + |m(s + 2\pi h_0)|^2 = 1$, $|m(s)|^2 + |m(s + 2\pi h_0)|^2 = 1$ and $m(s)m_\psi(s) + m(s + 2\pi h_0)m_\psi(s + 2\pi h_0) = 0$ hold. The reader can verify that the solution for $m_\psi(s)$ (in terms of $m(s)$) is of the form given in the proposition.

PROPOSITION 2.2. Let $\psi$ be an $A$-dilation MRA wavelet, then $e^{i\phi_0 u} |\hat{\psi}(s)|$ is the Fourier transform of an $A$-dilation MRA wavelet, where $u_1 = A^{-1} u$ and $u$ is the constant vector defined before Proposition 2.1.

PROOF. Let $\phi$ be the corresponding scaling function with low pass filter $m$, then $F^{-1}(\hat{\phi})$ is also an $A$-dilation scaling function whose associated low pass filter is $|m|$ by Lemma 2.3. Thus, the function $\psi_1$ defined by

\[
\hat{\psi_1}(A^s t) = e^{i\phi_0 u} |m(s + 2\pi h_0) \hat{\phi}(s)| = e^{i\phi_0 u} |\hat{\psi}(A^s t)|
\]

is an $A$-dilation MRA wavelet. The result follows after a simple substitution $t = A^s s$.

**3. Systems with Integrally Similar Dilation Matrices**

Two $d \times d$ integral matrices $B$ and $C$ are said to be integrally similar if there exists an integral $d \times d$ matrix $P$ such that $|\det(P)| = 1$ and $P^{-1} BP = C$. In such cases we write $B \sim C$. The main result of this section is the following theorem which reveals the relation between wavelets under integrally similar dilation matrices.
THEOREM 3.1. For any $2 \times 2$ integral matrix $P$ with $|\det P| = 1$, let $\Phi_P : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the operator defined by $\Phi_P(g(t)) = g(Pt)$. If $B$ and $C$ are two $2 \times 2$ integral, expansive matrices such that $P^{-1}BP = C$, then the following statements hold

(i) $\psi$ is a $B$-dilation wavelet iff $\Phi_P(\psi)$ is a $C$-dilation wavelet;

(ii) A function $f \in L^2(\mathbb{R}^2)$ is a $B$-dilation wavelet multiplier iff the function $\Phi_{(P^{-1})}(f)$ is a $C$-dilation wavelet multiplier.

Proof. (i) It suffices to show that $\{D^n_B(T^n\psi(t)) \ (n \in \mathbb{Z} \text{ and } \ell \in \mathbb{Z}^2)\}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$ if and only if $\{D^n_C(T^n\psi_C) \}$. By (i) above, there exists a $B$-dilation wavelet multiplier. On the other hand, if $D^n_B(T^n\psi(t)) = \{D^n_B(T^n\psi(t) = D^n_C(P(t-\ell)) \}
= |\det C|^n/2\psi(P(C^n-t-\ell)) |\det B|^n/2\psi((PC^nP^{-1}-P\ell) = |\det B|^n/2\psi(B^n(t-\ell)) = D^n_B(T^n\psi(t)) = D^n_C(T^n\psi_C(t))$ is an orthonormal basis of $L^2(\mathbb{R}^2)$.

(ii) Let $f$ be a $B$-dilation wavelet multiplier and let $\psi_C$ be a $C$-dilation wavelet. By (i) above, there exists a $B$-dilation wavelet $\psi$ such that $\psi_C(t) = \psi(Pt)$. We have

$$\widehat{\psi_C}(s) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \psi_C(t)e^{-ist}dt = \frac{1}{2\pi} \int_{\mathbb{R}^2} \psi(Pt)e^{-ist}dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \psi(t)e^{-iP^{-1}st}dt = \frac{1}{2\pi} \int_{\mathbb{R}^2} \psi(t)e^{-i(P\psi)^{-1}s}dt$$

Thus,

$$\mathcal{F}^{-1}(f_C\widehat{\psi_C})(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f((P\psi)^{-1}s)\widehat{\psi((P\psi)^{-1}s))e^{is\psi}ds$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} f(s)\widehat{\psi(s))e^{iP\psi}s}ds$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} f(s)\widehat{\psi(s))e^{i\psi}s}ds$$

$$= \mathcal{F}^{-1}(f\widehat{\psi})(Pt).$$

By the definition of $f$, $\mathcal{F}^{-1}(f\widehat{\psi})(t)$ is a $B$-dilation wavelet. Thus by (i) again, $\mathcal{F}^{-1}(f\widehat{\psi})(Pt)$ (hence $\mathcal{F}^{-1}(f_C\widehat{\psi_C})(t)$) is a $C$-dilation wavelet. This proves that $f_C$ is a $C$-dilation wavelet multiplier. On the other hand, if $f_C$ is a $C$-dilation wavelet multiplier, reversing the above argument shows that $f$ is a $B$-dilation wavelet multiplier.

Remark 3.1. The linear operator $\Phi_P : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined above is obviously continuous and unitary (since $|\det P| = 1$). In the case that $P$ is also integral and $P^{-1}BP = C$, then Theorem 3.1 asserts that $\Phi_P : W_B \to W_C$ is a continuous and bijective mapping, where $W_B$ is the set of all $B$-dilation wavelets and $W_C$ is the set of all $C$-dilation wavelets.
Using (2.3) the fact and the construction of the Haar-type integral matrices, we see that in the case \(B \sim C\) by the relation \(P^{-1}BP = C\), the operator \(\Phi_P\) is also a bijection between the set of all \(B\)-dilation MRA wavelets and the set of all \(C\)-dilation MRA wavelets.

We will now turn our focus on 2 \(\times\) 2 integral expansive matrices \(A\) with the property \(|\det(A)| = 2\). It turns out that there are exactly six integrally similar classes of such integral matrices \([12]\). A representative from each of these classes is listed below.

\[
A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}, A_3 = \pm \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, A_4 = \pm \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}.
\]

For the rest of this paper, we will only consider the case where \(A\) is one of the above six matrices. By Theorem 3.1 (as well as Remarks 3.1 and 3.2), the discussion of a different 2 \(\times\) 2 expansive integral matrix \(B\) (with \(|\det B| = 2\)) can be converted to a discussion concerning one of the six matrices listed above by applying the operator \(\Phi_P\) for some suitable \(P\). For our convenience, however, let us give the vectors \(h_0, u\) and \(A^{-1}u\) used in Propositions 2.1 and 2.2 here. We can choose \(u = (1, 0)^T\) for all cases. For \(A = A_1\) or \(A = A_2\), \(h_0 = (1/2, 0)^T\), \(u_1 = A^{-1}u = (0, 1/2)^T\); for \(A = \pm A_3\), \(h_0 = (1, 2, 1/2)^T\), \(u_1 = A^{-1}u = \pm (1/2, 1, 2)^T\); for \(A = \pm A_4\), \(h_0 = (1/2, 0)^T\) and \(u_1 = A^{-1}u = \pm (1/2, 1, 2)^T\). Throughout the rest of the paper, \(h_0, u\) and \(u_1\) are so defined with respect to their corresponding dilation matrix \(A\).

4. Examples of Haar and Shannon Type \(A\)-dilation Wavelets

Example 4.1. The construction of the Haar-type \(A\)-dilation wavelet given here can be found in \([3, 7, 12]\). The low pass filter \(m\) is given by \(m(s) = \frac{1}{2}(1 + e^{-i\pi u})\) and \(\hat{\phi}(s)\) is given by \(\hat{\phi}(s) = (1/2\pi) \prod_{j=1}^{\infty} m((A^\tau)^{-1}s)\). \(\psi\) is defined by

\[
\hat{\psi}(s) = e^{i\pi u_1} m((A^\tau)^{-1}s + 2\pi h_0) \hat{\phi}(s).
\]

Example 4.2. The Shannon type \(A\)-dilation MRA wavelet in this example is constructed using the concept of wavelet sets \([4, 5, 9]\). For each matrix \(A\), we construct a scaling set \(F\) such that the set \(E = A^\tau F \setminus F\) is an \(A\)-dilation wavelet set, i.e., the function \(\frac{1}{2\pi} \chi_{E}\) is the Fourier transform of an \(A\)-dilation wavelet. Let \(\Omega\) be the set \([-\pi, \pi]^2\). The low pass filter, scaling function and wavelet are given by

\[
m(s)|_\Omega = \chi_{(A^\tau)^{-1}\Omega}, \quad \hat{\phi}(s) = \frac{1}{2\pi} \chi_{\Omega} \text{ and } \hat{\psi}(s) = \frac{1}{2\pi} e^{i\pi u_1} \chi_{A^\tau\Omega\setminus\Omega}.
\]

Notice that \(m(s)\) is a \(2\pi\mathbb{Z}^2\)-translation periodic and the above formula gives its definition in one complete period (i.e., \(\Omega = [-\pi, \pi]^2\)). An illustration of the wavelet set \(E = A^\tau\Omega\setminus\Omega\) (which is the support of \(\hat{\psi}\)), as well as the supports of \(\hat{\phi}(s)\) (i.e., \(\Omega\)) and \(m(s)\) (within \(\Omega\)) are shown in Figures 1(a) to 1(c) for each case of \(A\).

Remark 4.1. In fact, the function \(\hat{\psi}_0(s) = \frac{1}{2\pi} \chi_{A^\tau\Omega\setminus\Omega}\) is itself the Fourier transform of an \(A\)-dilation MRA wavelet. From this fact, the above results on \(\psi\) can also be derived from Proposition 2.2 directly.
5. A-dilation Wavelet Multipliers

In this section, we characterize the A-dilation wavelet multipliers. This provides us an explicit way to obtain all A-dilation wavelet multipliers. A necessary condition for a function \( f \) to be an A-dilation wavelet multiplier is that \(|f| = 1\) [4, 13, 17]. Thus in the following we will limit our discussion to such functions. Instead of trying to characterize the scaling function multiplier or the low filter multiplier (which is the approach used in [13]), we will use a different approach. Let us call a function \( f \) with the property \(|f| = 1\) a unimodular function.

**Theorem 5.1.** A unimodular function \( f \in L^\infty(\mathbb{R}^2) \) is an A-dilation wavelet multiplier iff the function \( k(s) = f(A^s)/f(s) \) is \( 2\pi\mathbb{Z}^2 \)-translation periodic.

**Proof.** “\( \Rightarrow \)” Assume that \( f \in L^\infty(\mathbb{R}^2) \) is a unimodular function and that \( k(s) = f(A^s)/f(s) \) is \( 2\pi\mathbb{Z}^2 \)-translation periodic. To show that \( f \) is a wavelet multiplier, we need to show that for any A-dilation wavelet \( \psi, \eta = \mathcal{F}^{-1}(f\hat{\psi}) \) is also a wavelet. It suffices to verify that \( \hat{\eta} \) satisfies conditions (ii) and (iii) in Lemma 2.1. It is easy to see that (ii) holds for \( \hat{\eta} \) since \( |\hat{\eta}| = |\hat{\psi}| \) and (ii) holds for \( \hat{\psi} \). Applying the relation \( f(A^s) = k(s)f(s) \) repeatedly, for any \( j \geq 1 \) and \( \ell \in \mathbb{Z}^2 \), we obtain

\[
(5.1) \quad f((A^\tau)^j s) = k((A^\tau)^{-j} s) \cdots k(A^s)k(s)f(s),
\]

and

\[
f((A^\tau)^j(s + 2\pi\ell)) = k((A^\tau)^{-j-1}(s + 2\pi\ell))k((A^\tau)^{-j-2}(s + 2\pi\ell)) \cdots k(A^s(s + 2\pi\ell))k(s)f(s + 2\pi\ell) = k((A^\tau)^{-j} s) \cdots k(A^s)k(s)f(s + 2\pi\ell).
\]

Since \( k(s) \) is unimodular, this leads to

\[
f((A^\tau)^j s) \cdot f((A^\tau)^j(s + 2\pi\ell)) = k((A^\tau)^{-j} s) \cdots k(A^s)k(s)f(s) \cdot k((A^\tau)^{-j} s) \cdots k(A^s)k(s)f(s + 2\pi\ell) = f(s)f(s + 2\pi\ell)
\]
for any \( j \geq 0 \) and \( \ell \in \mathbb{Z}^2 \). Thus

\[
\sum_{j=0}^{\infty} \hat{n}((A^j)\ell) \hat{n}((A^j)(s + 2\pi\ell))
= \sum_{j=0}^{\infty} [f((A^j)\ell) f((A^j)(s + 2\pi\ell)) \cdot \hat{\psi}(A^j)(s) \hat{\psi}(A^j)(s + 2\pi\ell)]
= \sum_{j=0}^{\infty} f(s) f(s + 2\pi\ell) \hat{\psi}(A^j)(s) \hat{\psi}(A^j)(s + 2\pi\ell)
= f(s) f(s + 2\pi\ell) \sum_{j=0}^{\infty} \hat{\psi}(A^j)(s) \hat{\psi}(A^j)(s + 2\pi\ell) = 0
\]

for any \( \ell \in \mathbb{Z}^2 \setminus A^*\mathbb{Z}^2 \). So condition (iii) of Lemma 2.1 holds for \( \hat{n} \) as well.

“\( \Rightarrow \)” We need to show that \( k(s) = f(A^s) / f(s) \) is \( 2\pi \mathbb{Z}^2 \)-translation periodic. Let \( \psi \) be any \( A \)-dilation MRA wavelet such that \( \text{supp}(\hat{\psi}) = \mathbb{R}^2 \). Such \( \psi \) exists. For example the \( A \)-dilation wavelet constructed in Example 3.1 has such a property. By Proposition 2.2, the function \( \psi_1(\ell) \) defined by

\[
(5.2) \quad \hat{\psi}_1 = e^{i\lambda_0 u_1} |\hat{\psi}(s)| = e^{i\lambda_0 u_1} |\hat{\psi}(s)|
\]

is an \( A \)-dilation wavelet. Since \( \mathcal{F}^{-1}(\hat{\psi}_1) \) is also an \( A \)-dilation wavelet, \( \hat{\psi}_1 \) and \( f\hat{\psi}_1 \) both satisfy condition (iii) of Lemma 2.1, i.e.,

\[
(5.3) \quad \sum_{j=0}^{\infty} \hat{\psi}_1((A^j)\ell) \cdot \hat{\psi}_1((A^j)(s + 2\pi\ell)) = 0 \text{ a.e. and}
(5.4) \quad \sum_{j=0}^{\infty} f((A^j)\ell) \hat{\psi}_1((A^j)\ell) \cdot f((A^j)(s + 2\pi\ell)) \hat{\psi}_1((A^j)(s + 2\pi\ell)) = 0 \text{ a.e.}
\]

for any \( \ell \in \mathbb{Z}^2 \setminus A^*\mathbb{Z}^2 \). Since \( \ell \in \mathbb{Z}^2 \setminus A^*\mathbb{Z}^2 \), there exists \( \ell_1 \in \mathbb{Z}^2 \) such that \( \ell = \ell_0 + A^*\ell_1 = A^\ell (h_0 + \ell_1) \). It follows that \( \ell \circ u_1 = A^\ell (h_0 + \ell_1) \circ A^{-1} u = (h_0 + \ell_1) \circ u = 1/2 + m \) where \( h_0 \circ u \) by the definition of \( h_0 \) and \( u \), and \( m = \ell_1 \circ u \) is an integer. Thus

\[
\hat{\psi}_1(s) \hat{\psi}_1(s + 2\pi\ell) = e^{i\lambda_0 u_1} |\hat{\psi}_1(s)| \cdot e^{-i(s + 2\pi\ell)\circ u_1} |\hat{\psi}_1(s + 2\pi\ell)|
= e^{i(-\pi - 2\pi\ell)} |\hat{\psi}_1(s)| \cdot |\hat{\psi}_1(s + 2\pi\ell)| = -|\hat{\psi}_1(s)| \cdot |\hat{\psi}_1(s + 2\pi\ell)|.
\]

On the other hand, for any \( j > 0 \), \( (A^j)\ell \circ u_1 = \ell \circ A^{-j-1} u \in \mathbb{Z} \) hence

\[
\hat{\psi}_1((A^j)\ell) \hat{\psi}_1((A^j)(s + 2\pi\ell))
= e^{i(A^j)\lambda_0 u_1} |\hat{\psi}(A^j)(s)| \cdot e^{-i((A^j)(s + 2\pi\ell)\circ u_1)} |\hat{\psi}_1((A^j)(s + 2\pi\ell))|
= |\hat{\psi}_1((A^j)(s))| \cdot |\hat{\psi}_1((A^j)(s + 2\pi\ell))|
\]
Thus, (5.3) and (5.4) can be rewritten as

\[(5.5) \quad |\hat{\psi}_1(s)| \cdot |\hat{\psi}_1(s + 2\pi \ell)| = \sum_{j=1}^{\infty} |\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)| \] and

\[(5.6) \quad f(s)f((s + 2\pi \ell)) \cdot |\hat{\psi}_1(s)| \cdot |\hat{\psi}_1(s + 2\pi \ell)| = \sum_{j=1}^{\infty} f((A^\tau)^j s)f((A^\tau)^j s + 2\pi \ell))|\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)|].

Since \(f\) is unimodular, \(\overline{f} = 1/f\). Hence (5.6) can be rewritten as

\[(5.7) \quad \frac{f(s)}{f(s + 2\pi \ell)} |\hat{\psi}_1(s)| \cdot |\hat{\psi}_1(s + 2\pi \ell)| = \sum_{j=1}^{\infty} \frac{f((A^\tau)^j s)}{f((A^\tau)^j s + 2\pi \ell))}|\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)|].

Combining this with (5.5) then leads to

\[(5.8) \quad \sum_{j=1}^{\infty} |\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)| = \sum_{j=1}^{\infty} \frac{f(s + 2\pi \ell)}{f(s)} \frac{f((A^\tau)^j s)}{f((A^\tau)^j s + 2\pi \ell))}|\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)|].

Let \(\beta_j(s) = \frac{f(s + 2\pi \ell)}{f(s)} \frac{f((A^\tau)^j s)}{f((A^\tau)^j s + 2\pi \ell))}\), \(Re\beta_j(s) = a_j(s), \text{Im}\beta_j(s) = b_j(s)\), then (5.8) can be rewritten as

\[(5.9) \quad \sum_{j=1}^{\infty} (1 - a_j(s))|\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)| = i \sum_{j=1}^{\infty} b_j(s)|\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)|,

\[(5.10) \quad \sum_{j=1}^{\infty} (1 - a_j(s))|\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)| = 0 \text{ and}
\[(5.11) \quad \sum_{j=1}^{\infty} b_j(s)|\hat{\psi}_1((A^\tau)^j s)| \cdot |\hat{\psi}_1((A^\tau)^j s + 2\pi \ell)| = 0.

Since \(\beta_j\) is unimodular by its definition, \(a_j(s) \leq 1\). So we must have \(a_j(s) = 1\) a.e. in order for (5.10) to hold. Of course this would then imply that \(b_j(s) = 0\) a.e. as well since \(a_j^2(s) + b_j^2(s) = 1\). Thus,

\[\beta_j(s) = \frac{f(s + 2\pi \ell)}{f(s)} \frac{f((A^\tau)^j s)}{f((A^\tau)^j s + 2\pi \ell))} = 1, \text{ a.e.}\]

For \(j = 1\), the above is equivalent to

\[\frac{f(A^\tau s)}{f(s)} = \frac{f(A^\tau (s + 2\pi \ell))}{f(s + 2\pi \ell)} \text{ a.e. } \forall \ell \in \mathbb{Z}^2 \setminus A^\tau \mathbb{Z}^2,\]
i.e., \( k(s) = f(A^s) / f(s) \) is \( 2\pi \mathbb{Z}^2 \)-translation periodic.

Next, we show that all \( A \)-dilation wavelet multipliers can be constructed in the way described in the following theorem. Recall that an \( A \)-dilation wavelet set \( E \) in \( \mathbb{R}^2 \) is a measurable set such that \( \mathcal{F}^{-1}(\frac{1}{|\pi|} \chi_E) \) is an \( A \)-dilation wavelet. It is known that \( E \) is an \( A \)-dilation wavelet set iff both the sets \( \{ A^n E : n \in \mathbb{Z} \} \) and \( \{ E + 2\pi t : t \in \mathbb{Z}^2 \} \) are partitions of \( \mathbb{R}^2 \) modulo a null set \([5]\).

**Theorem 5.2.** Let \( E \) be an \( A \)-dilation wavelet set, \( k(s) \) be a measurable unimodular \( 2\pi \mathbb{Z}^2 \)-translation periodic function and \( g(s) \) be a measurable unimodular function defined on \( E \). Define

\[
f(s) = \begin{cases} 
g(s), & s \in E, \\
k((A^s)^{-1} \cdots ((A^s)^{-n} s) \cdot g((A^s)^{-n} s), & s \in (A^s)^n E, n \geq 1, \\
k(s)k((A^s)^{-1} \cdots ((A^s)^{-n} s) \cdot g((A^s)^{-n} s), & s \in (A^s)^{-n} E, n \geq 1, \\
1, & s = 0.
\end{cases}
\]

Then \( f \) is an \( A \)-dilation wavelet multiplier. Moreover, any \( A \)-dilation wavelet multiplier can be constructed this way.

**Proof.** Since \( k(s) \) is \( 2\pi \mathbb{Z}^2 \)-translation periodic, it suffices to show that \( f(A^s) = k(s) \cdot f(s) \) in order to show that \( f \) is an \( A \)-dilation wavelet multiplier by Theorem 5.1.

**Case 1.** \( s \in E \). Then \( A^s \in A^s E \) and

\[
f(A^s) = k((A^s)^{-1} A^s)g((A^s)^{-1} A^s) = k(s)g(s) = k(s)f(s).
\]

**Case 2.** \( s \in (A^s)^n E \) where \( n \geq 1 \). Then \( A^s \in (A^s)^{n+1} E \) and

\[
f(A^s) = k((A^s)^{-1} A^s) \cdots k((A^s)^{-(n+1)} A^s)g((A^s)^{-(n+1)} A^s) \\
= k(s)k((A^s)^{-1} A^s) \cdots k((A^s)^{-n} A^s)g((A^s)^{-n} A^s) \\
= k(s)f(s).
\]

**Case 3.** \( s \in (A^s)^{-1} E \). Then \( A^s \in E \) and \( f(s) = \overline{k(s)}g(A^s) \), so \( f(A^s) = g(A^s) = k(s)f(s) \).

**Case 4.** \( s \in (A^s)^{-n} E \) where \( n > 1 \). Then \( A^s \in (A^s)^{-(n-1)} E \) and

\[
f(A^s) = k(A^s) \cdots k((A^s)^{-n+1} A^s)g((A^s)^{-n+1} A^s) \\
= k(s)k(A^s) \cdots k((A^s)^{-n+1} A^s)g((A^s)^{-n+1} A^s) \\
= k(s)f(s).
\]

Since \( \{ (A^s)^n E : n \in \mathbb{Z} \} \) is a partition of \( \mathbb{R}^2 \) modulo a null set, the above four cases have exhausted all possibilities for \( s \in \mathbb{R}^2 \) in the a.e. sense.

Now suppose that \( f(s) \) is an \( A \)-dilation wavelet multiplier. Let \( g(s) = f(s) \) for \( s \in E \), and \( k(s) = f(A^s) / f(s) \). Then \( k(s) \) is \( 2\pi \mathbb{Z}^2 \)-translation periodic and is unimodular. We leave it to our reader to verify that \( f(s) \) has the form given in the theorem. In other word, any \( A \)-dilation wavelet multiplier can be constructed this way. \( \square \)
6. Phases of $A$-dilation MRA Wavelets

The linear phase filtering problem is considered in signal processing where wavelets and scaling functions are considered as filter functions. For more detailed discussions on the linear-phase problems concerning wavelet and scaling functions, interested reader may refer to [2, Section 5.5].

A function $f(t) \in L^2(\mathbb{R}^2)$ is said to have a linear phase if its Fourier transform has the form

$$ \hat{f}(s) = \pm |\hat{f}(s)| \cdot e^{-is\cdot a} \text{ a.e.} $$

for some constant vector $a \in \mathbb{R}^2$, which is the phase of $\hat{f}(s)$.

The following theorem concerning the phase of an $A$-dilation MRA wavelet in $L^2(\mathbb{R}^2)$ is an application of the results obtained in the last section.

**Theorem 6.1.** Let $\psi(t) \in L^2(\mathbb{R}^2)$ be an $A$-dilation MRA wavelet, then

$$ \hat{\psi}(s) = e^{is\cdot u} f(s) \hat{\phi}(s) $$

for some $A$-dilation wavelet multiplier $f(s)$.

**Proof.** By Proposition 2.1, the Fourier transform of an $A$-dilation MRA wavelet $\psi(t)$ has the form

$$ \hat{\psi}(s) = e^{is\cdot u} v(s) m((A^s)^{-1}s + 2\pi h_0) \hat{\phi}((A^s)^{-1}s), $$

where $v$ is some unimodular and $2\pi \mathbb{Z}^2$-translation periodic function. Recall that the low pass filter $m(s)$ is $2\pi \mathbb{Z}^2$-translation periodic and

$$ \hat{\phi}(A^s s) = m(s) \hat{\phi}(s). $$

Let $\hat{\phi}(s) = g(s) \hat{\phi}(s)$ and $E = \text{Supp}(\hat{\phi})$. On the one hand, we have

$$ \hat{\phi}(A^s s) = g(A^s s) \hat{\phi}(A^s s) = g(A^s s) m(s) \hat{\phi}(s). $$

On the other hand, we also have

$$ \hat{\phi}(A^s s) = m(s) \hat{\phi}(s) = m(s) g(s) \hat{\phi}(s). $$

It follows that $g(A^s s)/g(s) = m(s)/|m(s)|$. Since $m(s)$ is $2\pi \mathbb{Z}^2$-translation periodic, this means that $g(A^s s)/g(s)$ is $2\pi \mathbb{Z}^2$-translation periodic on its support. By (6.2), $(A^s)^{-1} E \subset E$, so $\{(A^s)^n E : n \in \mathbb{Z}\}$ is an increasing sequence of sets. We now define a $2\pi \mathbb{Z}^2$-translation periodic function $k(s)$ for $s \in \mathbb{R}^2$ based on $g$ in the following steps.

Step 1. For $s \in E \cap (A^s)^{-1} E = (A^s)^{-1} E$, define $k(s) = g(A^s s)/g(s)$.

Step 2. For $s \in \bigcup_{t \in \mathbb{Z}^2} ((A^s)^{-1} E + 2\pi t) \setminus (A^s)^{-1} E$, define $k(s) = k(t)$ if $s = t + 2\pi \ell_0$ for some $t \in (A^s)^{-1} E$ and $\ell_0 \in \mathbb{Z}^2$. $k(s)$ is well-defined since we have shown earlier that $g(A^s s)/g(s)$ is $2\pi \mathbb{Z}^2$-translation periodic on its support.

Step 3. For $s \in \mathbb{R}^2 \setminus \bigcup_{t \in \mathbb{Z}^2} ((A^s)^{-1} E + 2\pi t)$, define $k(s) = 1$.

By its definition, $k(s)$ is unimodular and $2\pi \mathbb{Z}^2$-translation periodic. We will now extend the definition of $g$ from $E$ to $\mathbb{R}^2$. First, for $s \in A^s E \setminus E$, define $g(s) = k((A^s)^{-1} s) \cdot g((A^s)^{-1}s)$. Then for $s \in (A^s)^2 E \setminus A^s E$, define $g(s) = k((A^s)^{-1} s) \cdot g((A^s)^{-1}s)$. In general, $g$ is defined on $(A^s)^n E$ ($n > 0$) inductively. The support of $\psi$ is contained in $A^s E$ by (6.1). By Lemma 2.1(ii), $\bigcup_{n \in \mathbb{Z}} (A^s)^n E = \mathbb{R}^2$ modulo a
null set. Thus the extended $g$ has been defined on the entire $\mathbb{R}^2$. $g$ is an $A$-dilation wavelet multiplier since $k(s) = g(A^s)/g(s)$ is unimodular and $2\pi\mathbb{Z}^2$-translation periodic. For $s \in (A^*)^{-1}E$, we have $m(s) = \hat{\phi}(A^s)/\hat{\psi}(s) = k(s)|m(s)|$. For $s \in E \setminus (A^*)^{-1}E$, $\hat{\phi}(A^s) = m(s)\hat{\phi}(s) = 0$ while $\hat{\phi}(s) \neq 0$. So $m(s) = 0$. Thus $m(s) = k(s)|m(s)|$ also holds. This means $m(s) = k(s)|m(s)|$ holds for all $s \in \mathbb{R}^2$ since $m(s)$ is $2\pi\mathbb{Z}^2$-translation periodic and $\bigcup_{k \in \mathbb{Z}^2} (E + 2\pi k) = \mathbb{R}^2$ modulo a null set by Lemma 2.3(i). Finally, (6.1) becomes

$$
\hat{\psi}(s) = e^{is\omega_0} v(s) k((A^*)^{-1}s + 2\pi h_0) m((A^*)^{-1}s + 2\pi h_0) |g((A^*)^{-1}s)|
$$

Let $f(s) = v(s)k((A^*)^{-1}s + 2\pi h_0)g((A^*)^{-1}s)$. Since the support for each of $v(s)$, $k(s)$ and $g(s)$ is $\mathbb{R}^2$, the support for $f(s)$ is $\mathbb{R}^2$. Furthermore,

$$
\begin{align*}
\frac{f(A^s)}{f(s)} &= \frac{v(A^s)k(s + 2\pi h_0)g(s)}{v(s)k((A^*)^{-1}s + 2\pi h_0)g((A^*)^{-1}s)} \\
&= (v(A^s)/v(s))k(s + 2\pi h_0)k((A^*)^{-1}s + 2\pi h_0)k((A^*)^{-1}s).
\end{align*}
$$

Since $(v(A^s)/v(s))k(s + 2\pi h_0)$ is $2\pi\mathbb{Z}^2$-translation periodic by the definitions of $v$ and $k$, we only need to show that $k((A^*)^{-1}s + 2\pi h_0)k((A^*)^{-1}s)$ is also $2\pi\mathbb{Z}^2$-translation periodic. If $\ell \in A^*\mathbb{Z}^2$ then it is obvious that $k((A^*)^{-1}(s + 2\pi\ell) + 2\pi h_0)k((A^*)^{-1}(s + 2\pi\ell)) = k((A^*)^{-1}s + 2\pi h_0)k((A^*)^{-1}s)$. Otherwise, we have $\ell = \ell_0 + A^*\ell_1 = A^*(h_0 + \ell_1)$ for some $\ell_1 \in \mathbb{Z}^2$. It follows that $k((A^*)^{-1}(s + 2\pi\ell) + 2\pi h_0)k((A^*)^{-1}(s + 2\pi\ell)) = k((A^*)^{-1}s + 2\pi\ell_1 + 4\pi h_0)k((A^*)^{-1}s + 2\pi\ell_1 + 2\pi h_0) = k((A^*)^{-1}s + 2\pi h_0)k((A^*)^{-1}s)$ since $2h_0 \in \mathbb{Z}^2$. This proves that $f(A^s)/f(s)$ is indeed $2\pi\mathbb{Z}^2$-translation periodic. $f(A^s)/f(s)$ is unimodular since every term in the right side of (6.3) is unimodular. Thus $f$ is an $A$-dilation wavelet multiplier by Theorem 5.1. 

**Corollary 6.1.** For every $A$-dilation wavelet $\psi$, there exists an $A$-dilation wavelet $\psi'$ such that $|\hat{\psi}| = |\hat{\psi}'|$ and $\psi'$ has a linear phase $-u_1 = -A^{-1}u$.

**Proof.** By Theorem 6.1, there exists an $A$-dilation wavelet multiplier $f$ such that

$$
\hat{\psi}(s) = e^{is\omega_0} f(s) |\hat{\psi}(s)|.
$$

Since $f$ is unimodular, multiplying $\overline{f}$ on both sides of the above equation yields

$$
\overline{f(s)}\hat{\psi}(s) = e^{is\omega_0} |\hat{\psi}(s)| = e^{is\omega_0} |\overline{f(s)}\hat{\psi}(s)|.
$$

Since $\overline{f}$ is itself an $A$-dilation wavelet multiplier, $\psi'$ defined by $\hat{\psi}'(s) = \overline{f(s)}\hat{\psi}(s)$ is also an $A$-dilation wavelet. By definition, $-u_1$ is a linear phase of $\psi'$. 

**Remark 6.1.** If $B$ is a $2 \times 2$ integral expansive matrix with $|\det B| = 2$ and $P^{-1}AP = B$ for some integral matrix $P$ with $|\det P| = 1$, then for any given $B$-dilation wavelet $\psi_B$, there exists a $B$-dilation wavelet $\psi'_B$ such that $|\hat{\psi}_B| = |\hat{\psi}'_B|$ and $\psi'_B$ has a linear phase of the form $-P^*u_1$. 


7. Path-connectivity of the Set of $A$-dilation MRA Wavelets

As another application of Theorem 5.1, in this section we prove that the set of $A$-dilation MRA wavelets is path-connected under the $L^2(\mathbb{R}^2)$ norm topology. For more discussions and related results on this topic, interested reader may refer to [4, 10, 14, 16, 17]. Our main result of this section is the following theorem.

**Theorem 7.1.** For any two $A$-dilation MRA wavelets $\psi_0$ and $\psi_1$, there exists a continuous map $\gamma : [0, 1] \rightarrow L^2(\mathbb{R}^2)$ such that $\gamma(0) = \psi_0$, $\gamma(1) = \psi_1$ and $\gamma(t)$ is an $A$-dilation MRA wavelet for all $t \in [0, 1]$.

We will prove the theorem by directly constructing a continuous path connecting the two MRA wavelets. The proof is given for the case where $A$ is one of the matrices $A_1$, $A_2$, $\pm A_3$ and $\pm A_4$. In general, if $B \sim A$ for one of the matrices $A$ above, then we can simply apply the unitary operator $\Phi$ to the set of all $A$-dilation MRA wavelets (recall Remark 3.2). The proof is of constructive nature and consequently is long. So we break it into several lemmas. For a given $A$-dilation wavelet $\psi_0$, define $\mathcal{M}_{\psi_0} = \{\psi : \hat{\psi} = \hat{\psi_0}\}$ for some $A$-dilation wavelet multiplier $\psi$, and $\mathcal{W}_{\psi_0} = \{\psi : \psi$ is an $A$-dilation wavelet with $|\hat{\psi}| = |\hat{\psi_0}|\}$. Furthermore, in the case that $\psi_0$ is an $A$-dilation MRA wavelet with $\phi_0$ being the corresponding $A$-dilation scaling function for the MRA, define $\mathcal{S}_{\psi_0} = \{\psi : \psi$ is an $A$-dilation MRA wavelet with $|\phi| = |\phi_0|\}$.

**Lemma 7.1.** $\mathcal{S}_{\psi_0} = \mathcal{M}_{\psi_0} = \mathcal{W}_{\psi_0}$ for any $A$-dilation MRA wavelet $\psi_0$.

**Proof.** $\mathcal{W}_{\psi_0} \subseteq \mathcal{S}_{\psi_0}$ follows from equation (2.4) of Lemma 2.4. $\mathcal{M}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$ by definition. $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$ follows from an argument similar to the one used in the proof of [13, Theorem 1.2] and Proposition 2.1. \hfill $\square$

**Lemma 7.2.** Let $\psi_0$ be an $A$-dilation MRA wavelet. Then $\mathcal{M}_{\psi_0}$ is path-connected.

**Proof.** This is proved in [13] for a special case of $A$. However the proof for the general case is similar and is omitted. \hfill $\square$

By Lemma 7.1 $\mathcal{S}_{\psi_0} = \mathcal{M}_{\psi_0}$. Thus, to show that any two $A$-dilation MRA wavelets are connected by a continuous path, it suffices to show that for any $A$-dilation MRA wavelet $\psi$, there exists a $\psi_1 \in \mathcal{S}_{\psi}$, such that $\psi_1$ is path-connected to the generalized Shannon wavelet $\psi_0$ defined by

$$\psi_0(s) = \frac{1}{2\pi}e^{i\omega_0} \chi_{\Omega}(s).$$

We will choose $\psi_1 \in \mathcal{S}_{\psi}$ so that it is associated with a scaling function $\phi_1$ such that $\phi_1 \geq 0$ and $m_1 \geq 0$ and

$$\psi_1(s) = e^{i\omega_1} m_1((A^*)^{-1}s + 2\pi h_0)\phi_1((A^*)^{-1}s).$$

The existence of such a $\psi_1$ is guaranteed by Lemma 2.3 and Proposition 2.2. Note that the corresponding scaling function and low pass filter of $\psi_0$ are given by is $\phi_0(s) = (1/2\pi) \chi_{\Omega}$ and $m_0(s)|_{\Omega} = \chi_{(A^*)^{-1}\Omega}$.

We will now build a path that connects the low-pass filters first, then use it to construct the path for the scaling functions and the wavelet functions. The
construction shown is for the case of $A = A_3$, the other cases can be similarly treated. Notice in this case $2\pi b_0 = (\pi, \pi)^T$ and $s \circ u_1 = (s_1 + s_2)/2$ where $s = (s_1, s_2)^T$. For $t \in [0,1]$, $s \in \Omega = [-\pi, \pi]^2$, define

$$m_t(s) = \begin{cases} 
(1-t)m_0(s) + tm_1(s), & s \in (A^T)^{-1}\Omega \setminus (1-t)(A^T)^{-1}\Omega \\
1, & s \in (1-t)(A^T)^{-1}\Omega \\
\sqrt{1 - |m_t(s + (\pi,-\pi)^T)|^2}, & s \in R_1 \\
\sqrt{1 - |m_t(s + (\pi,\pi)^T)|^2}, & s \in R_2 \\
\sqrt{1 - |m_t(s + (\pi,-\pi)^T)|^2}, & s \in R_3 \\
\sqrt{1 - |m_t(s + (\pi,\pi)^T)|^2}, & s \in R_4,
\end{cases}$$

where the regions $R_j$ ($1 \leq j \leq 4$) are as marked in Figure 1(b). The general $m_t(s)$ is then defined by extending the above periodically so that it is a $2\pi Z^2$-periodic function. Of course, for $t = 0$ and $t = 1$, $m_t(s)$ is just the $m_0(s)$ and $m_1(s)$ given before. Furthermore, it is easy to see that $|m_t(s)| \leq 1$ for any $t$ by its definition and that $m_t(s)$ satisfies the equation

$$|m_t(s)|^2 + |m_t(s + (\pi,\pi)^T)|^2 = 1.$$

Define:

$$(7.3) \quad \hat{\phi}_t(s) = \frac{1}{2\pi} \prod_{j=1}^{\infty} m_t(((A^T)^{-1}s)^j)$$

$$(7.4) \quad \hat{\psi}_t(s) = e^{i\frac{2\pi t}{\pi}} m_t((A^T)^{-1}s + (\pi,\pi)^T)\hat{\phi}_t((A^T)^{-1}s)$$

for $s \in \mathbb{R}^2$. $\hat{\phi}_t$ is well defined since $0 \leq m_t(s) \leq 1$, so is $\hat{\psi}_t$. Furthermore, for $t = 0$ and $t = 1$, $\psi_t$ coincides with the $\hat{\psi}_0$ and $\hat{\psi}_1$ defined in (7.1) and (7.2) respectively.

To complete the proof of Theorem 7.1, we need to show

1. $\phi_t$ is an $A$-dilation scaling function, so $\psi_t$ is an $A$-dilation MRA wavelet.
2. The mapping $[0,1] \to L^2(\mathbb{R}^2)$ defined by $t \mapsto \psi_t$ is continuous.

These statements are implied by the next three lemmas so they will follow once these lemmas are proved.

**Lemma 7.3.** For each $t \in [0,1]$, $\phi_t$ (as defined in (7.3)) is an $A$-dilation scaling function for some $A$-dilation MRA hence the corresponding $\psi_t$ in (7.4) is an $A$-dilation MRA wavelet.

**Proof.** The statement holds trivially for $t = 0$ and 1, so we only need to consider the case $0 < t < 1$. From the definition of $\phi_t$, we have

$$(7.5) \quad \hat{\phi}_t(A^Ts) = m_t(s)\hat{\phi}_t(s), \ s \in \mathbb{R}^2;$$

$$(7.6) \quad \hat{\phi}_t(s) = \frac{1}{2\pi}, \ s \in (1-t)\Omega.$$

So $\hat{\phi}_t(s)$ satisfies conditions (ii) and (iii) of Lemma 2.3. We will prove that $\phi_t$ satisfies condition (i) of Lemma 2.3 as well, which then implies that $\phi_t$ is a scaling function.

For $\forall \ s \in \Omega$, $(A^T)^{-j}s \in (A^T)^{-1}\Omega$, $\forall \ j \geq 1$. So by the definition of $m_t(s)$, we have $m_t((A^T)^{-j}s) \geq 1 - t$. Since $A$ is expansive, for any fixed $0 < t < 1$, we can
choose \( k_0 \) sufficiently large such that for \( k \geq k_0 \), \((A^r)^{-k} \Omega \subset (1-t) \Omega \). Hence, if \( s \in \Omega \) and \( k \geq k_0 \), then \( \hat{\phi}_t((A^r)^{-k}s) = 1/2\pi \) by (7.6) and

\[
\hat{\phi}_t(s) = \frac{1}{2\pi} \prod_{j=1}^{\infty} m_t((A^r)^{-j}s) = \frac{1}{2\pi} \prod_{k=1}^{k_0} m_t((A^r)^{-k}s) \prod_{k=k_0+1}^{\infty} m_t((A^r)^{-k}s)
\]

\[= \hat{\phi}_t((A^r)^{-k_0}s) \prod_{k=1}^{k_0} m_t((A^r)^{-k}s) = \frac{1}{2\pi} \prod_{k=1}^{k_0} m_t((A^r)^{-k}s) \geq \frac{1}{2\pi} (1-t)^{k_0}.
\]

This implies that \( \chi_\Omega(s) \leq 2\pi \hat{\phi}_t(s)/(1-t)^{k_0} \). Define

\[
\mu_{t,k}(s) = \frac{1}{2\pi} \chi_\Omega((A^r)^{-k}s) \prod_{j=1}^{k} m_t((A^r)^{-j}s), \; k \geq 1.
\]

Then

\[
\mu_{t,k}(s) \leq \frac{\hat{\phi}_t((A^r)^{-k}s)}{(1-t)^{k_0}} \prod_{j=1}^{k} m_t((A^r)^{-j}s) = \frac{\hat{\phi}_t(s)}{(1-t)^{k_0}}.
\]

For \( k \geq 2 \), we have

\[
\int_{\mathbb{R}^2} |\mu_{t,k}(s)|^2 e^{-i\omega s} ds
\]

\[= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left| \chi_\Omega((A^r)^{-k}s) \right|^2 \prod_{j=1}^{k} |m_t((A^r)^{-j}s)|^2 e^{-i\omega s} ds
\]

\[= \frac{2^k}{4\pi^2} \int_{\Omega} \prod_{j=1}^{k} |m_t((A^r)^{k-j}s)|^2 e^{-i\omega((A^r)^{k}s)} ds
\]

\[= \frac{2^k}{4\pi^2} \int_{\Omega} \prod_{j=0}^{k-1} |m_t((A^r)^j s)|^2 e^{-i\omega((A^r)^{k}s)} ds
\]

\[= \frac{2^k}{4\pi^2} \int_{\Omega} |m_t(s)|^2 \prod_{j=1}^{k-1} |m_t((A^r)^j s)|^2 e^{-i\omega((A^r)^{k}s)} ds.
\]

Let \( R_j \) and \( T_j \) (\( 1 \leq j \leq 4 \)) be the regions marked in Figure 1(b) and let \( U_j = R_j \cup T_j \). To compute the last integral in the above equality, we divide \( \Omega \) into these smaller regions. We have

\[
\int_{\mathbb{R}^2} |\mu_{t,k}(s)|^2 e^{-i\omega s} ds
\]

\[= \frac{2^k}{4\pi^2} \int_{\bigcup_{j \leq j' \leq 4} U_{j'}} |m_t(s)|^2 \prod_{j=1}^{k-1} |m_t((A^r)^j s)|^2 e^{-i\omega((A^r)^{k}s)} ds
\]

\[= \frac{2^k}{4\pi^2} \sum_{j' \leq 1} \int_{U_{j'}} |m_t(s)|^2 \prod_{j=1}^{k-1} |m_t((A^r)^j s)|^2 e^{-i\omega((A^r)^{k}s)} ds.
\]
We have
\[
\int_{U_1} |m_t(s)|^2 \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds = \int_{R_1} |m_t(s)|^2 \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds
\]
\[
+ \int_{T_1} |m_t(s)|^2 \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds
\]
\[
= \int_{T_1} \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds,
\]
where the second equality is obtained by letting \( s = s - (\pi, -\pi)^T \) in the integral over \( T_1 \) together with the equality that \( |m_t(s)|^2 + |m_t(s + (\pi, -\pi)^T)|^2 = 1 \) for any \( s \in R_1 \). Similarly, for each \( j' = 2, 3 \) and 4 we also have
\[
\int_{U_{j'}} |m_t(s)|^2 \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds
\]
\[
= \int_{T_{j'}} \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds
\]
So
\[
\int_{\mathbb{R}^2} |\mu_{t,k}(s)|^2 e^{-i\omega s} ds
\]
\[
= \frac{2^k}{4\pi^2} \int_{\cup_{j'=1}^{4} T_{j'}} \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds
\]
\[
= \frac{2^k}{4\pi^2} \int_{(A^T)^{-1} \Omega} \prod_{j=1}^{k-1} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^k s)} ds
\]
\[
= \frac{2^{k-1}}{4\pi^2} \int_{\Omega} \prod_{j=0}^{k-2} |m_t((A^T)^j s)|^2 e^{-i\omega((A^T)^{k-1} s)} ds
\]
\[
= \int_{\mathbb{R}^2} |\mu_{t,k-1}(s)|^2 e^{-i\omega s} ds.
\]
Repeating the above procedure then leads to
\[
\int_{\mathbb{R}^2} |\mu_{t,k}(s)|^2 e^{-i\omega s} ds = \int_{\mathbb{R}^2} |\mu_{t,1}(s)|^2 e^{-i\omega s} ds
\]
\[
= \frac{1}{4\pi} \int_{\mathbb{R}^2} |\chi_\Omega((A^T)^{-1} s)|^2 \cdot |m_t((A^T)^{-1} s)|^2 e^{-i\omega s} ds
\]
\[
= \frac{2}{4\pi^2} \int_\Omega |m_t(s)|^2 e^{-i\omega(A^T) s} ds
\]
\[
= \frac{2}{4\pi^2} \int_{(A^T)^{-1} \Omega} e^{-i\omega(A^T) s} ds = \frac{1}{4\pi^2} \int_\Omega e^{-i\omega s} ds = \delta_{\omega,0}.
\]
So \( \|\mu_{t,k}\|^2 = 1 \). Clearly \( \lim_{k \to \infty} \mu_{t,k}(s) = \widehat{\phi}_t(s) \) for all \( s \in \mathbb{R}^2 \). Thus \( \phi_t \in L^2(\mathbb{R}^2) \) by Fatou's Lemma. Since \( \mu_{t,k}(s) \) is dominated by \( \frac{\widehat{\phi}_t(s)}{(1-t)^m} \),

\[
\lim_{k \to \infty} \int_{\mathbb{R}^2} |\mu_{t,k}(s)|^2 e^{-im\phi(s)} ds = \int_{\mathbb{R}^2} |\widehat{\phi}_t(s)|^2 e^{-im\phi(s)} ds = \delta_{n,0}
\]

by Lebesgue's dominated convergence theorem. This is equivalent to \( \sum_{t \in \mathbb{Z}^2} |\widehat{\phi}_t(s + 2\pi t)|^2 = \frac{1}{4\pi^2}, \ a.e. \) By Lemma 2.3, \( \phi_t \) is a scaling function for some MRA. Consequently, \( \psi_t \) is an A-dilation MRA wavelet.

**Lemma 7.4.** \( \lim_{t \to t_0} \widehat{\phi}_t(s) = \widehat{\phi}_{t_0}(s) \) a.e. for any \( t_0 \in [0,1] \).

**Proof.** By the definition of \( m_t(s) \), the mapping \( t \mapsto m_t(s) \) is continuous with respect to \( t \) a.e. for \( s \in \mathbb{R}^2 \). Since \( \widehat{\phi}_1 \geq 0 \), \( \lim_{j \to -\infty} \widehat{\phi}_1((A^t)^{-j} s) = 1/2\pi \) a.e. For any given \( \varepsilon > 0 \) and \( s \in \mathbb{R}^2 \), there exists a positive integer \( n_0 \) such that \( \widehat{\phi}_t((A^t)^{-n} s) > 1/2\pi - \varepsilon/2 \) and \( (A^t)^{-n} s \subset (A^t)^{-1} \Omega \) for any \( n \geq n_0 \). It follows that \( m_t((A^t)^{-n} s) \) is either 1 or \( (1 - t) + tm_t((A^t)^{-n} s) \) for any \( t \in [0,1] \). In either case, \( m_t((A^t)^{-n} s) \geq m_1((A^t)^{-n} s) \), so the following inequality holds for any \( t \in [0,1] \):

\[
\widehat{\phi}_t((A^t)^{-n} s) = \frac{1}{2\pi} \prod_{j=1}^\infty m_t((A^t)^{-j} (A^t)^{-n} s) \\
\geq \frac{1}{2\pi} \prod_{j=1}^\infty m_1((A^t)^{-j} (A^t)^{-n} s) = \widehat{\phi}_1((A^t)^{-n} s).
\]

Since \( \widehat{\phi}_t(s) \leq 1/2\pi \) for any \( s' \in \mathbb{R}^2 \) by its definition, it follows that for any \( t_1, t_2 \in [0,1] \), we have

\[
(7.7) \quad |\widehat{\phi}_{t_1}((A^t)^{-n} s) - \widehat{\phi}_{t_2}((A^t)^{-n} s)| < \varepsilon/2.
\]

On the other hand, since \( t \mapsto m_t((A^t)^{-j} s) \) is continuous for each \( j \), so is \( t \mapsto \prod_{j=1}^{n_0} m_t((A^t)^{-j} s) \). Hence for each \( t_0 \in [0,1] \), there exists \( \delta > 0 \) such that for \( |t - t_0| < \delta \) and \( t \in [0,1] \),

\[
\left| \prod_{j=1}^{n_0} m_t((A^t)^{-j} s) - \prod_{j=1}^{n_0} m_{t_0}((A^t)^{-j} s) \right| < \varepsilon.
\]

Now,

\[
|\widehat{\phi}_{t}(s) - \widehat{\phi}_{t_0}(s)| = \left| \prod_{j=1}^{n_0} m_t((A^t)^{-j} s) - \prod_{j=1}^{n_0} m_{t_0}((A^t)^{-j} s) \right|
\]

\[
= \left| \prod_{j=1}^{n_0} m_t((A^t)^{-j} s) \widehat{\phi}_{t}((A^t)^{-n_0} s) - \prod_{j=1}^{n_0} m_{t_0}((A^t)^{-j} s) \widehat{\phi}_{t_0}((A^t)^{-n_0} s) \right|
\]

\[
= \left| \prod_{j=1}^{n_0} m_t((A^t)^{-j} s) \cdot \widehat{\phi}_{t}((A^t)^{-n_0} s) - \prod_{j=1}^{n_0} m_{t_0}((A^t)^{-j} s) \widehat{\phi}_{t_0}((A^t)^{-n_0} s) \right|
\]

\[
+ \left| \prod_{j=1}^{n_0} m_{t_0}((A^t)^{-j} s) \widehat{\phi}_{t}((A^t)^{-n_0} s) - \prod_{j=1}^{n_0} m_{t_0}((A^t)^{-j} s) \widehat{\phi}_{t_0}((A^t)^{-n_0} s) \right|
\]

\[
\leq \frac{1}{2\pi} \prod_{j=1}^{n_0} m_t((A^t)^{-j} s) - \prod_{j=1}^{n_0} m_{t_0}((A^t)^{-j} s) \right| + \left| \widehat{\phi}_{t}((A^t)^{-n_0} s) - \widehat{\phi}_{t_0}((A^t)^{-n_0} s) \right|
\]

\[
< \frac{\varepsilon}{2\pi} + \frac{\varepsilon}{2} < \varepsilon.
\]
For what non-integral expansive matrices (if any) that allow \( A \) to be expansive matrices \( A \mapsto t \mapsto r > 1 \) that do not satisfy the condition \(|\det A| = 2\)?

\[ \lim_{t \to t_0} \hat{\psi}_{t_0}(s) = \hat{\psi}_{t_0}(s). \]

By the continuity of \( m_t(s) \) and \( \hat{\psi}_t \), we know that \( \lim_{t \to t_0} \hat{\psi}_{t}(s) = \hat{\psi}_{t_0}(s) \), a.e.

**Lemma 7.5.** For \( t_0, t \in [0, 1] \), \( \lim_{t \to t_0} \|\hat{\psi}_{t} - \hat{\psi}_{t_0}\|^2 = 0. \)

**Proof.** Since \( \|\hat{\psi}_{t}\|^2 = \|\hat{\psi}_{t_0}\|^2 = 1 \), \( \|\hat{\psi}_{t} - \hat{\psi}_{t_0}\|^2 = \langle \hat{\psi}_{t} - \hat{\psi}_{t_0}, \hat{\psi}_{t} - \hat{\psi}_{t_0} \rangle = 2 - \langle \hat{\psi}_{t}, \hat{\psi}_{t_0} \rangle - \langle \hat{\psi}_{t_0}, \hat{\psi}_{t} \rangle \). Thus it suffices to show that \( \lim_{t \to t_0} \hat{\psi}_{t}, \hat{\psi}_{t_0} \) is continuous. This completes the proof of the connectedness theorem.

Since \( \hat{\psi}_{t_0} \in L^2(\mathbb{R}^2) \), for any given \( \varepsilon > 0 \), there exists a sufficiently large number \( r > 0 \) such that \( (\int_{|s| > r} |\hat{\psi}_{t_0}(s)|^2 ds)^{1/2} < \varepsilon/4 \). By Hölder Inequality, we then have

\[ \int_{|s| > r} |\hat{\psi}_{t}(s) - \hat{\psi}_{t_0}(s)| \cdot |\hat{\psi}_{t_0}(s)| ds \leq \|\hat{\psi}_{t}(s) - \hat{\psi}_{t_0}(s)\| \cdot (\int_{|s| > r} |\hat{\psi}_{t_0}(s)|^2 ds)^{1/2} < \varepsilon/2 \]

since \( |\hat{\psi}_{t}(s)| \leq 1/2\pi \) and \( |\hat{\psi}_{t_0}(s)| \leq 1/2\pi \) by (7.3), (7.4) and the fact that \(|m_{s}| \leq 1\) for any \( t \). Thus by the dominated convergence theorem, we have \( \lim_{t \to t_0} \int_{|s| > r} |\hat{\psi}_{t}(s) - \hat{\psi}_{t_0}(s)| ds = 0 \). Therefore, there exists a number \( \delta > 0 \) that \( \int_{|s| > r} |\hat{\psi}_{t}(s) - \hat{\psi}_{t_0}(s)| ds < \pi \varepsilon/2 \) whenever \(|t - t_0| < \delta\) Combining the above leads to

\[ \left| \langle \hat{\psi}_{t}, \hat{\psi}_{t_0} \rangle - 1 \right| \leq \left| \langle \hat{\psi}_{t}, \hat{\psi}_{t_0} \rangle - \langle \hat{\psi}_{t_0}, \hat{\psi}_{t_0} \rangle \right| = \left| \int_{\mathbb{R}^2} (\hat{\psi}_{t}(s) - \hat{\psi}_{t_0}(s)) \cdot \hat{\psi}_{t_0}(s) ds \right| \]

\[ \leq \int_{|s| \leq r} |(\hat{\psi}_{t}(s) - \hat{\psi}_{t_0}(s))| \cdot \hat{\psi}_{t_0}(s) ds + \int_{|s| > r} |(\hat{\psi}_{t}(s) - \hat{\psi}_{t_0}(s))| ds < \varepsilon. \]

So \( \lim_{t \to t_0} \|\hat{\psi}_{t} - \hat{\psi}_{t_0}\|^2 = 0. \)

Since the inverse Fourier transform is continuous, we know that the mapping \( t \mapsto \psi_{t} \) is continuous. This completes the proof of the connectedness theorem.

**8. Some Open Questions**

We end our paper with some unsolved problems related to wavelet multipliers.

**Question 8.1.** Let \( A \) be a \( 2 \times 2 \) expansive matrix that is not necessarily integral or with the property \(|\det A| = 2\), characterize \( f \in L^2(\mathbb{R}^2) \) such that \( F^{-1}(f\psi) \) is an \( A \)-dilation wavelet whenever \( \psi \) is.

In the case \( d = 2 \), it is known that \( A \)-dilation wavelets exist for integral and expansive matrices \( A \) that do not satisfy the condition \(|\det A| = 2\) \([4, 5]\). One example of such matrices \( A \) is \( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \). If we do not require that \( A \) being integral, then again there exist single function \( A \)-dilation wavelets for some such matrices.

An example in the case \( d = 2 \) is \( \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \) \([4, 5]\).

**Question 8.2.** For what non-integral expansive matrices (if any) that allow single function \( A \)-dilation wavelets, there exists a related MRA structure? In the case that single function \( A \)-dilation MRA wavelets exist for a non-integral expansive matrix \( A \), can we characterize the \( A \)-dilation wavelet multipliers?
In general, if $A$ is a $d \times d$ integral expansive matrix with $|\det A| = 2$ (and $d > 2$), are there any $A$-dilation MRA wavelets? Can we characterize the $A$-dilation wavelet multipliers for such matrices? Notice that for the case $d = 3$, there are infinite many such integral expansive matrices. For instance,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

is one such, as one can check. In general, we suspect that this is true for any $d > 2$.

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**References**


