Realizable Powers of Ropelengths by Nontrivial Knot Families

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Abstract. For any given knot \( K \), a thick realization \( K_0 \) of a knot type \( K \) is a knot of unit thickness which is of the knot type \( K \). The ropelength of \( K \) is defined as the arc length of the shortest thick realization of \( K \). A recent result shows that there exists a constant \( b > 0 \) such that for any knot type \( K \), its ropelength \( L(K) \) is bounded above by \( b \cdot (Cr(K))^{3/2} \), where \( Cr(K) \) is the crossing number of \( K \). It is also known that there exists a family of infinitely many knot types \( \{K_n\} \) such that \( n = Cr(K_n) \to \infty \) as \( n \to \infty \) and \( L(K_n) = O(n) \). In this paper, we show that for each \( p \) with \( 3/4 \leq p \leq 1 \), there exists a family of infinitely many knot types \( \{K_n\} \) with the property that \( a_0 \cdot (Cr(K_n))^p \leq L(K_n) \leq b_0 \cdot (Cr(K_n))^p \), where \( a_0 \) and \( b_0 \) are some positive constants. In other word, any power between 3/4 and 1 is realizable by some knot family.

1. Introduction

In this paper, we are interested in the relation between the crossing number of a thick knot and its arc length. The thickness \( \tau(K) \) of a smooth space curve \( K \) is defined, in this paper, to be the supremal radius of the embedded normal tubes around \( K \). See [CKS2], [DER1] and [LSDR] (among others) for the various definitions of thickness. For any given knot \( K \), a thick realization \( K_0 \) of \( K \) is a knot of unit thickness (i.e. \( \tau(K) = 1 \)) which is of the same knot type as \( K \). Let \( L(K) \) be the length of the smooth space curve \( K \). The ropelength \( L(K) \) of a knot type \( K \) is the infimum of \( L(K) \) taken over all thick realizations of \( K \). The existence of \( L(K) \) is shown in [CKS2]. Note that we may also use \( L(K) \) for the length of a knot \( K \). In case that \( K \) is a smooth knot with unit thickness and of knot type \( K \), then we have \( L(K) \geq L(K) \). We also note that knots realized on the cubic lattice are sometimes useful since they have a naturally defined length and thickness. The above definition of rope length can be made using any of the different thickness definitions, and the results obtained using one definition generally also hold for the other thickness definitions modulo a suitable scale change which will only effect the constants in the results.

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For the lower bound of ropelength, it is shown in [B] and [BS] that the minimum crossing number of a knot of unit thickness is bounded by a constant times its rope length to the four-thirds power. In other word, for any knot $K$ with unit thickness, we have $L(K) \geq c \cdot (Cr(K))^{3/4}$ for some constant $c$. The constant $c$ is estimated to be at least 1.105 by the result obtained in [BS] and is improved to 2.135 recently ([RS]). This four third power is also shown to be achievable for torus knot families ([CKS1],[DE]). That is, there exists a family of knot types $\{K_n\}$, such that $L(K_n) \leq c_0 \cdot (Cr(K))^3$ for some constant $c_0 > 0$. On the other hand, for the upper bound of ropelength, it has been shown in [DEY] recently that there exists a constant $d > 0$ such that $L(K) \leq d \cdot (Cr(K))^{3/2}$ for any knot type $K$. While it is still an open question whether this $3/2$ power is realizable by any knot family, it has been shown in [DET] that there exists a family of prime knot types $\{K_n\}$ with the property that $Cr(K_n) \rightarrow \infty$ (as $n \rightarrow \infty$) such that $L(K_n)$ grows linearly with respect to $Cr(K_n)$.

A power $p$ is called realizable by a knot family $\mathcal{K}_n$ (with $Cr(\mathcal{K}_n) \rightarrow \infty$) if $L(\mathcal{K}_n) = O((Cr(\mathcal{K}_n))^p)$. Thus, any power $p < 3/4$ or $p > 3/2$ is not realizable by any knot family and $p = 3/4$ and $p = 1$ are realizable by some knot families. In this paper, we will explore the gap between the power $3/4$ and 1, that is, whether any power $p$ with $3/4 \leq p \leq 1$ is realizable by an infinite family of knot types $\{\mathcal{K}_n\}$. More specifically, we will prove that for each such number $p$, there exist a family of infinitely many knot types $\{\mathcal{K}_n\}$ with the property that $Cr(\mathcal{K}_n) \rightarrow \infty$ (as $n \rightarrow \infty$) such that $L(\mathcal{K}_n)$ grows linearly with respect to $(Cr(\mathcal{K}_n))^p$. However, in the case that a power $q > 1$ is realizable by another knot family, our method cannot be applied directly to show that all the powers between 1 and $q$ are also realizable. Furthermore, the knot families exhibited in this paper with the desired properties are all composite knots of two prime factors. So the problem, if the powers of $p$, for $3/4 \leq p \leq 1$ are realizable using prime knots remains open at this time.

2. Preliminary results

In this section, we will list some known results that we will need in the next section. Notice that some of these are classical knot theory results and we list them here mainly for the sake of convenience of the reader.

**Theorem 2.1.** [Mu], [DE] Let $m \geq 2$ be a positive integer, then the $(m^2, m^2 + 1)$ torus knot $T_n$ has crossing number $n = m^3 - 1$ and $L(T_n) = O(n^{3/4})$. In particular, there exists a lattice representation $T_n$ of $T_n$ such that $L(T_n) \leq 16m^3$.

**Theorem 2.2.** [S1], [S2] For any knot $K$, let $b(K)$ be its bridge number and let $g(K)$ be its genus. Let $K_1 \# K_2$ be the connected sum of the knots $K_1$ and $K_2$, then $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$ and $g(K_1 \# K_2) = g(K_1) + g(K_2)$.

**Theorem 2.3.** [M], [S2] The genus of an $(a, b)$ torus knot $T(a, b)$ is $g(T(a, b)) = \frac{(a-1)(b-1)}{2}$ and the bridge number of $T(a, b)$ is $b(T(a, b)) = \min(a, b)$.

**Theorem 2.4.** [Bu], [DET] Let $M_n$ denote the knot whose Conway symbol is $(3, 3, \ldots, 3, 2)$ $(n \geq 2)$ (as shown in Figure 2). Then $Cr(M_n) = 3n + 2$, $b(M_n) = n + 1$ and $L(M_n) = O(n)$. Furthermore, the genus of $M_n$ is $n + 1$. 
Remark 2.1. Figure 1 shows a standard diagram of $M_n$. It is easy to see that the knot $M_n$ shown in the figure can indeed be realized with $O(n)$ steps in the cubic lattice. Thus for $M_n$ on the cubic lattice the ratio $\frac{\text{length}}{\text{crossing number}}$ is bounded by some constant $c$. This example is a particular type of Montesinos knot. A Montesinos knot is a knot (or link) obtained by concatenating $r$ rational tangles, as shown in Figure 2 for the case of $r = 3$. In general $r$ can be any integer greater or equal to two. The label $\frac{\beta_i}{\alpha_i}$, $1 \leq i \leq r$ stands for the rational tangle classified by that same fraction $\frac{\beta_i}{\alpha_i}$. A treatment of Montesinos knots can be found in [BuZ]. Assuming that $r \geq 3$ and that for all $i$, $\alpha_i \geq 2$, it is shown in [BZ] that for any such Montesinos knot $M$ we have the bridge number $b(M) = r$. The example in Figure 1 is a Montesinos knot with maximal $r$ for a given crossing number $n$. While the above knot $M_n$ is alternating, Montesinos knots are not necessarily alternating in general since tangles of different signs can be substituted into the diagram in Figure 2.

Figure 1. The knot $M_n$

Figure 2. The diagram of a Montesinos knot with $r = 3$

Theorem 2.5. [Liv] The genus $g$ of a Seifert surface of a non-trivial knot or link $K$ constructed from a projection diagram $D$ of $K$ is bounded above by $(2 + c - s - \ell)/2$, where $s$ is the number of Seifert circles in $D$, $\ell$ is the number of components of $K$, and $c$ is the number of crossings in $D$. In particular, for a one component knot $K$, we have $Cr(K) \geq 2g(K) + 1$ since $\ell = 1$ and $s \geq 2$ for any $D$ when $K$ is a non-trivial knot.
Theorem 2.6. [DET] Let $K$ be a knot and let $b(K)$ be its bridge number. If $P$ is a polygonal knot of the same knot type as $K$ on the cubic lattice, then the length $L(P)$ of $P$ is at least $6b(K)$. On the other hand, if $K$ is of unit thickness, then $L(K) \geq 2\pi b(K)$.

We end this section with one last (rather obvious) observation: If $P$ is a polygonal knot of the knot type $K$ on the cubic lattice, then $L(K)$ is bounded above by $2L(P)$, where $L(P)$ is the length of $P$.

3. The main theorem and its proof

We are now ready to state and prove the main theorem of this article:

Theorem 3.1. For any real number $p$ with $3/4 \leq p \leq 1$, there exists a family of knots $\{K_n\}$ with the property that $Cr(K_n) \to \infty$ (as $n \to \infty$) such that $L(K_n) = O((Cr(K_n))^p)$.

Proof. The construction of the knot types $K_n$ uses a connected sum of the Montesinos knots defined in Theorem 2.4 and of the $(m^2, m^2 + 1)$ torus knots in Theorem 2.1. Let $p$ be any positive constant between $3/4$ and $1$. Let $K_{n1}$ be the $(m^2, m^2 + 1)$ torus knot with $m = \lceil n^{1/4p}\rceil$. Then $Cr(K_{n1}) = m^4 - 1$, $b(K_{n1}) = m^2$ and $g(K_{n1}) = (m^2 - 1)m^2/2$ by Theorems 2.1 and 2.3. Let $K_{n2}$ be the $(3, 3, \ldots, 3, 2)$ Montesinos knot. We have $Cr(K_{n2}) = 3n + 2$, $g(K_{n2}) = n + 1$ and $b(K_{n2}) = n + 1$ by Theorem 2.4. Let $K_n = K_{n1} \# K_{n2}$. By Theorem 2.2, $g(K_n) = m^2(m^2 - 1)/2 + n + 1$. Therefore, by Theorem 2.5,

\[
Cr(K_n) \geq 2g(K_n) + 1 = m^2(m^2 - 1) + 2n + 3 = O(m^4) = O(n^{1/p})
\]

and $Cr(K_n)^p = O(n)$. Moreover

\[
b(K_n) = n + m^2 > n.
\]

By Theorem 2.6, we have $L(K_n) \geq 2\pi n = O((Cr(K_n))^p)$. This shows that $L(K_n)$ is at least of the order of $O((Cr(K_n))^p)$.

A lattice representation $P$ of $K_{n2}$ can be seen in Figure 3 and from this we can estimate that $L(P) \leq 34n + 26$. Using Theorem 2.1 we can find a lattice representation $K_n = K_{n1} \# K_{n2}$ such that $L(K_n) \leq 16m^3 + 34n + 26 = O(n) = O((Cr(K_n))^p)$. It follows that $L(K_n)$ is at most of the order of $O((Cr(K_n))^p)$ by the observation at the end of the last section. This finishes the proof of the theorem. $\square$

Remark 3.1. Notice that the knot $K_n$ so constructed is a composite knot of two prime knots. $K_n$ can also be constructed using a variety of other knots and links. For example, $K_{n2}$ can be replaced by any other torus knots and links whose ropelengths follow the $3/4$-power law [B]. The knot $K_{n1}$ can be replaced by a different Montesinos knot consisting of $n + 1$ rational tangles or with prime links that look like medieval chain mail which trivially must have a linear relationship between their lengths on the lattice and their crossing numbers.

Remark 3.2. It is a long standing conjecture that $Cr(K_1 \# K_2) = Cr(K_1) + Cr(K_2)$. In many cases it is not even possible to show that $Cr(K_1 \# K_2) \geq$
Cr(K_i) for i = 1 or 2. It has just been proved by one of the authors that Cr(K_1#K_2) = Cr(K_1) + Cr(K_2) if K_1 and K_2 are both torus knots [D]. Furthermore, Cr(K_1#K_2) ≥ Cr(K_1) + 3 for any torus knot K_1 and nontrivial knot K_2. This result also holds when the torus knot is replaced by the knot M_n. So in fact we do have Cr(T(a, b)#M_n) = Cr(T(a, b)) + Cr(M_n).

Remark 3.3. In [DET], a question is raised about whether the ropelength of any alternating knot must be bounded below by its crossing number times some constant. This question has just been answered in the negative by J. Cantarella and C. Mullikin recently. They reported that the power 3/4 is realizable by the T(2, n) family [CM]. Notice that T(2, n) is an alternating knot for any odd n ≥ 3.

4. Further discussions

We end this paper by offering the following observations and questions.

A. Our proof of the main theorem relies on the crucial fact that (up to a suitable scalar) the bridge number of a knot bounds its ropelength from below. It is obvious that the bridge number can not be the only factor affecting the ropelength of a knot since there are knots with a fixed bridge number and arbitrarily high crossing number. In fact, if a power q > 1 is realizable by a knot family, then the bridge number as used in Theorem 2.6 cannot serve as a lower bound for the ropelength since the bridge number of a knot is bounded above by its crossing number. This also means that given a realizable q > 1, the method used here will not apply directly to show that all powers between 1 and this power q are also realizable by some knot families.

B. Can we prove that all powers between 3/4 and 1 are realizable by families of prime knots?
C. What powers are realizable by the torus knot families?

References


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