The Crossing Numbers of Thick Knots and Links

Yuanan Diao and Claus Ernst

ABSTRACT. In this paper, we derive a closed integral formula for the modified average crossing number of thick knots and links. We also strengthen some known inequalities between the crossing number of a thick knot or thick link with its arc-lengths.

1. Introduction

In mathematics, a knot is typically defined as a simple closed curve that is volumeless. Under such definition, no matter how complex a knot may be, it can always be realized by a curve of fixed arc length. But when a knot is to be realized by a physical subject such as a rope, the volume of the subject has to be taken into consideration. Given a piece of rope with a fixed length, one can only tie certain knots with it, depending on the length and flexibility of the rope. To model a knot with such physical properties, the concept of knot thickness is introduced, see [LSDR] or [DER1] for example. This concept can be generalized trivially to the case of links. Once we have defined the thickness of a knot, we can then ask the basic question: what kind of knots (with unit thickness) can be tied with a given length of curve? This is a very challenging problem. Since the crossing number is a good measure of the complexity of a knot, it is natural to ask the following simpler question: how big can the crossing number of a knot with unit thickness be given its arc length?

For the sake of simplicity, we will adopt the thickness defined in [LSDR] (which is defined for $C^2$ curves but can be extended to $C^{1,1}$ curves [CKS2]) throughout this paper. Let $K$ be a knot of unit thickness, $L(K)$ be its arc length and $Cr(K)$ be its crossing number. The following inequality is obtained in [BS]:

\begin{equation}
Cr(K) \leq \frac{11}{4\pi} \left( L(K) \right)^{\frac{3}{4}},
\end{equation}

or equivalently,

\begin{equation}
L(K) \geq \left( \frac{4\pi Cr(K)}{11} \right)^{\frac{4}{3}}.
\end{equation}

1991 Mathematics Subject Classification. Primary 57M25.

Key words and phrases. Knots, links, crossing number, thickness of knots, arc length of knots.
This means that if we know the crossing number of $K$, then we know a lower bound of the arc length of $K$. On the other hand, if we know the length of $K$, then we know an upper bound of $Cr(K)$. In [DE], it is shown that there exists a family of knots (with unit thickness) such that $Cr(K) \geq 0.046(L(K))^{\frac{2}{3}}$. This shows that the four third power in (1.1) is sharp. Similar result is obtained in [B1],[CKS1]. For links, the Gaussian linking number is an alternative measure of link complexity. Let $(K_1, K_2)$ be a two-component link (of unit thickness) with $K_1$ and $K_2$ be the two components. Let $L_1$ and $L_2$ be the arc length of $K_1$ and $K_2$ respectively. Also, let $Lk(K_1, K_2)$ be the linking number of $(K_1, K_2)$. It is shown in [DER2] that there exists a constant $c > \frac{26}{\pi}$ such that

$$Lk(K_1, K_2) < \min \{cL_1^{\frac{2}{3}}, cL_2^{\frac{2}{3}}\}.$$  

Similar result holds when $K_1$ and $K_2$ are polygonal knots on the unit cubic lattice. These results revealed a key asymptotic relation between the length of a knot (or link) and the maximal crossing number (or linking number) it may have. While improvement is certainly possible, examples show that it can only be done in the coefficient, not the power. In the second half of this paper we will derive sharper upper bounds for the constants in these inequalities obtained in [BS], [D] and [DER2].

Notice that for knots with relatively small crossing number or short arc length, (1.2) is not very informative. In such cases, the following formula (also obtained in [BS]) is more effective:

$$Cr(K) \leq \frac{1}{16\pi}L^2(K) \text{ or } L(K) \geq 4\sqrt{\pi \cdot Cr(K)}.$$  

For example, when $Cr(K) = 3$, (1.2) only produces a lower bound of 2.51 for $L(K)$, while (1.4) yields a lower bound of 12.27 for $L(K)$. For $Cr(K) = 20$ (a crossing number beyond the current available knot tabulation tables), we are getting $L(K) > 10.45$ from (1.2) and $L(K) > 31.70$ from (1.4). For small knots, getting a larger lower bound for $L(K)$ is not an easy task. In fact, for a long time, we could not even answer the following seemingly easy question: can one tie a knot with one foot of one-inch diameter rope? In other word, if the arc length of a knot with unit thickness is less than or equal to 24, can it be a non-trivial knot [LSDR]? The reason that it seems easy is that experiments and computer simulations ([B2],[P]) suggest that the minimum rope length required to tie a non-trivial knot (with a unit thickness rope) is around 32. Some progress has been made in this direction lately. First, it is proven in [CKS2] that within the set of all $C^1$ knot of unit thickness of any given knot type, there exists one that has the minimal arc-length (called the rope-length minimizer. It is shown in [LSDR] that the arc length of a knot with unit thickness is at least $5\pi \approx 15.71$. This lower bound has been improved to $2\pi(2 + \sqrt{2}) \approx 21.45$ [CKS2]. The following formula obtained in [D] further improves this result.

$$Cr(K) \leq \frac{L(K)(L(K) - 17.334)}{16\pi},$$  

or equivalently,

$$L(K) \geq \frac{1}{2} \left(17.334 + \sqrt{17.334^2 + 64\pi Cr(K)}\right).$$
Table 1 below shows the numerical results produced by (1.6) for $Cr(K) \leq 20$. This is by far the best known lower bound for $L(K)$ for knots with small crossing numbers. It shows that if $K$ is a knot more complicated than the trefoil, then the arc length of $K$ has to be at least 25.285. In the case that $K$ is a trefoil, with some extra effort, it is shown that $L(K) > 24$ indeed [D]. Some other results regarding small links are also obtained in [CKS2].

<table>
<thead>
<tr>
<th>$Cr(K) =$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(K) &gt;$</td>
<td>23.69</td>
<td>25.28</td>
<td>26.73</td>
<td>28.07</td>
<td>29.33</td>
<td>30.51</td>
<td>31.63</td>
<td>32.70</td>
<td>33.72</td>
</tr>
<tr>
<td>$Cr(K) =$</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>$L(K) &gt;$</td>
<td>34.71</td>
<td>35.65</td>
<td>36.57</td>
<td>37.46</td>
<td>38.32</td>
<td>39.15</td>
<td>39.97</td>
<td>40.76</td>
<td>41.53</td>
</tr>
</tbody>
</table>

A key concept used in [D] is the modified average crossing number of a thick knot $K$ (denoted by $m(K)$). In this paper, we will first derive a closed integral formula for $m(K)$. We will then use this formula and other known methods to obtain sharper upper bounds for the constants in these inequalities mentioned earlier.

2. The Modified Averaging Crossing Number

The concepts and some results (with their proofs) in this section are developed in [D]. We include them here for the sake of convenience to our reader. The discussions in this section applies to any knot $K$ that has a regular projection to a plane (or to a sphere under spherical projection) with the exception that the projection may contain a few points such that each of them is the projection of a line segment in $K$. For any two points $Y$ and $Z$ on $K$, there are two arcs of $K$ joining these two points. To distinguish them, we give $K$ a fixed orientation as follows: Assume that $K$ is given by a parametric equation, then the orientation is naturally defined by the increasing direction of the parameter. The arc from $Y$ to $Z$ following this orientation is denoted by $\alpha(Y, Z)$ and the arc from $Z$ to $Y$ is denoted by $\alpha(Z, Y)$. Let $P(A)$ be the projection of any subset $A$ of $K$ to the plane or sphere ($A$ may be a point, an arc or $K$ itself). If $Y, Z \in K$ and $P(Y) = P(Z)$, then $P(\alpha(Y, Z))$ is a loop (possibly with self-intersections) in $P(K)$. Let $\overline{YZ}$ be the line segment joining $Y$ and $Z$ and consider the following two simple closed curves. One of them is $K_1 = \overline{YZ} \cup (K \setminus \alpha(Y, Z))$. The other one is $Q_1 = \overline{YZ} \cup \alpha(Y, Z)$. If $Q_1$ is a trivial knot and $\alpha(Y, Z)$ can be deformed to $\overline{YZ}$ via an ambient isotopy that is identity on $K \setminus \alpha(Y, Z)$, then $K_1$ is ambient isotopic to $K$. We will say that $\alpha(Y, Z)$ is replaceable by $\overline{YZ}$ (or simply replaceable) and that $P(\alpha(Y, Z))$ is a removable loop in $P(K)$. Replacing $\alpha(Y, Z)$ with $\overline{YZ}$ is called a replacing operation. A replacing operation reduces the total number of crossings in $P(K)$ without creating any new crossings or moving any remaining crossings (and their corresponding points on $K$). Figure 1 shows some replaceable loops in $P(K)$. Observe that if $P(\alpha(Y, Z))$ is a loop in $P(K)$ without self intersections such that all passes of $P(K)$ intersecting the loop overpass it (or all underpass it), then $\alpha(Y, Z)$ is replaceable by $\overline{YZ}$. To see this, one needs to consider $P(\alpha(Y, Z))$ in $P(K)$. A sequence of suitable Reidemeister moves can be carried out to reduce $P(\alpha(Y, Z))$ under the given condition. A case of this is shown in Figure 1(a).
Since a replacing operation reduces the total number of crossings of $P(K)$ by at least one, it can only be repeated finitely many times. When there are no more removable loops left, we come to a new knot $K'$, which is ambient isotopic to $K$. The crossings of $P(K')$ and the corresponding points on $K'$ projected to these crossings are subsets of $P(K)$ and $K$ respectively. We call $K'$ a weakly reduced knot of $K$ and $P(K')$ a weakly reduced knot diagram of $K$. $K'$ may not be obtained by just eliminating the original removable loops in $P(K)$ since removing a removable loop may create new ones. $K'$ is not unique in general either since it depends on where the replacing process starts and which loops we choose to eliminate first. The number of crossings in $P(K')$ may not be the minimal crossing number of the knot type $K$. This is why we call $K'$ a weakly reduced knot of $K$.

The following is a key lemma proved in [D] and we refer to its proof there.

**Lemma 2.1.** Let $K$ be a knot with thickness one and $P$ be a regular plane projection of $K$ to a plane. Then there exists a weakly reduced knot $K'$ of $K$ such that any strong under-over point pair $Y$ and $Z$ in it must satisfy the condition $\rho(Y, Z) \geq 2\pi$ (and $\rho(Z, Y) \geq 2\pi$). The same conclusion holds for a spherical projection.

Let $\vec{v}$ be a unit vector and let $\Sigma_{\vec{v}}$ be the plane normal to $\vec{v}$ that passes through the origin. When we project two arcs $\ell'$ and $\ell''$ to $\Sigma_{\vec{v}}$, they may intersect each other and we can count the number of crossings in the projection. We will use $c_{\vec{v}}(\ell', \ell'')$ to denote the number of crossings between $\ell'$ and $\ell''$ under the projection to $\Sigma_{\vec{v}}$. By symmetry, we have $c_{\vec{v}}(\ell', \ell'') = c_{\vec{v}}(\ell'', \ell')$. The averaging crossing number between $\ell'$ and $\ell''$ is then defined as

\begin{equation}
(2.1) \quad a(\ell', \ell'') = \frac{1}{4\pi} \int_S c_{\vec{v}}(\ell', \ell'')d\mu,
\end{equation}

where $S$ is the unit sphere centered at the original point and $\mu$ is the measure of the area on $S$. Similarly, we will use $c_{\vec{v}}(K)$ to denote the number of crossings in

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{removable_loops.png}
\caption{Examples of Removable Loops}
\end{figure}
the projection of $K$ to $\Sigma_\vec{v}$ (via direction $\vec{v}$). The average crossing number of $K$ is defined as

\[ a(K) = \frac{1}{4\pi} \int_S c_\vec{v}(K) d\mu. \]  

(2.2)

Let $\gamma_1$ and $\gamma_2$ be two arc-length parameterized equations of $\ell$ and $\ell''$ respectively. It is shown in [FH] that

\[ a(\ell', \ell'') = \frac{1}{4\pi} \int_{\ell'} \int_{\ell''} + \int_{\ell''} \int_{\ell'} \frac{|(\dot{\gamma}_1(t), \dot{\gamma}_2(s), \gamma_1(t) - \gamma_2(s))|}{|\gamma_1(t) - \gamma_2(s)|^3} dt ds. \]  

(2.3)

Similarly,

\[ a(K) = \frac{1}{4\pi} \int_K \int_K \frac{|(\dot{\gamma}(t), \dot{\gamma}(s), \gamma(t) - \gamma(s))|}{|\gamma(t) - \gamma(s)|^3} dt ds, \]  

(2.4)

where $\gamma$ is an arc-length parameterized equation of $K$.

Let $n$ be a large positive integer and divide $K$ into $n$ arcs of equal length so that each arc is of length $L(K)/n$. Number them according to the orientation of $K$ as $\ell_1, \ell_2, \ldots, \ell_n$. We have

\[ c_\vec{v}(K) = \frac{1}{2} \sum_{1 \leq i, j \leq n} c_\vec{v}(\ell_i, \ell_j), \]  

(2.5)

hence

\[ a(K) = \frac{1}{4\pi} \int_S c_\vec{v}(K) d\mu = \frac{1}{2} \sum_{1 \leq i, j \leq n} a(\ell_i, \ell_j). \]  

(2.6)

The factor $\frac{1}{2}$ is in there because if a crossing is counted in $a(\ell_i, \ell_j)$, it will be counted again in $a(\ell_j, \ell_i)$. Let $Cr(K)$ be the minimum crossing number of $K$. Notice that this is the minimum of all possible projections of all knots that are of the same knot type as $K$. So we have $Cr(K) \leq c_\vec{v}(K)$ for any $\vec{v}$ hence $Cr(K) \leq a(K)$. Assume that $K$ has unit thickness, then by Lemma 2.1, if we only count the crossings $P(Y) = P(Z)$ in $P(K)$ that satisfy the conditions $\rho(Y, Z) \geq 2\pi$ and $\rho(Z, Y) \geq 2\pi$ (such a number is uniquely determined by the projection), we still get a number that is at least $Cr(K)$. The number of crossings counted this way is called the modified crossing number of $K$ (in the projection to $\Sigma_\vec{v}$) and is denoted by $m_\vec{v}(K)$. Of course, we have $m_\vec{v}(K) \leq c_\vec{v}(K)$. The modified average crossing number of $K$, denoted by $m(K)$, is then defined by

\[ m(K) = \frac{1}{4\pi} \int_S m_\vec{v}(K) d\mu. \]  

(2.7)

(2.6) can be modified to:

\[ m(K) \approx \frac{1}{2} \sum_{\rho(\ell_i, \ell_j) > 2\pi - \frac{L(K)}{2}} a(\ell_i, \ell_j) \approx \frac{1}{2} \sum_{1 \leq i \leq n} \sum_{\ell_j \in A_i} a(\ell_i, \ell_j), \]  

(2.8)
where \( J_i \) is the set of points on \( K \) that are of \( 2\pi \) arc-length distance away from \( \ell_i \). Applying (2.3) to (2.8), we get
\[
m(K) \approx \frac{1}{2} \sum_{1 \leq i \leq n} \sum_{\ell_j \in J_i} a(\ell_i, \ell_j)
\]
(2.9)
\[
\approx \sum_{1 \leq i \leq n} \frac{1}{4\pi} \int_{\ell_i} \int_{J_i} \frac{|(\dot{\gamma}(t), \dot{\gamma}(s), \gamma(t) - \gamma(s))|}{|\gamma(t) - \gamma(s)|^3} dt ds.
\]
Let \( n \to \infty \), we then obtain the following theorem.

**Theorem 2.2.** Let \( K \) be a knot with unit thickness and arc-length parameterized equation \( \gamma \). Then we have
\[
m(K) = \frac{1}{4\pi} \int_K \int_{K_s} \frac{|(\dot{\gamma}(t), \dot{\gamma}(s), \gamma(t) - \gamma(s))|}{|\gamma(t) - \gamma(s)|^3} dt ds,
\]
where \( K_s \) is the set of points \( \gamma(t) \) on \( K \) satisfying the condition \( |t - s| \geq 2\pi \), that is, the set of points that are \( 2\pi \) arc-length distance away from \( \gamma(s) \).

In the case that \( K \) is of thickness \( T \neq 1 \), it is not hard to show that the formula corresponding to (2.10) is
\[
m(K) = \frac{1}{4\pi} \int_K \int_{K_s} \frac{|(\dot{\gamma}(t), \dot{\gamma}(s), \gamma(t) - \gamma(s))|}{|\gamma(t) - \gamma(s)|^3} dt ds,
\]
where \( K_s \) is the set of points \( \gamma(t) \) on \( K \) satisfying the condition \( |t - s| \geq 2\pi T \).

### 3. The Crossing Number of a Thick Knot

In this section, we improve the inequalities (1.1) or (1.5). Our approach is modified from that used in [BS]. The reader is encouraged to read the reference [BS] to better understand the argument here.

In order to estimate \( m(K) \) using (2.10), we need to bound
\[
\frac{1}{4\pi} \int_{K_s} \frac{|(\dot{\gamma}(t), \dot{\gamma}(s), \gamma(t) - \gamma(s))|}{|\gamma(t) - \gamma(s)|^3} dt
\]
for each fixed \( s \). By the properties of the triple scalar multiple and the fact that \( \dot{\gamma}(t) \) and \( \dot{\gamma}(s) \) are unit vectors, it follows that \( |(\dot{\gamma}(t), \dot{\gamma}(s), \gamma(t) - \gamma(s))| \leq |\gamma(t) - \gamma(s)| \).

Thus
\[
\int_{K_s} \frac{|(\dot{\gamma}(t), \dot{\gamma}(s), \gamma(t) - \gamma(s))|}{|\gamma(t) - \gamma(s)|^3} dt \leq \int_{K_s} \frac{1}{|\gamma(t) - \gamma(s)|^2} dt.
\]

Let \( B_r(X) \) be the (open) ball of radius \( r \) centered at \( X = \gamma(s) \). Following the notations in [BS], we will define \( B_s(a, b) = B_o(\gamma(s)) \setminus B_o(\gamma(s)) \) and \( K_s(a, b) = K_s \cap B_s(a, b) \). We will also let \( \ell(a, b) \) be the total arc-length of \( K_s(a, b) \). As mentioned in [BS], for any segment \( A \) of \( K \), the solid tube \( t(A) \) about \( A \) is the union of all unit disks normal to \( K \) at some point \( y \in A \). The volume of \( t(A) \) is simply \( \pi \cdot \ell(A) \) where \( \ell(A) \) is the total arc length of \( A \). Since \( K_s \) does not intersect \( B_2(X) \) in its interior, we have \( \ell(0, 2) = 0 \). Let us now estimate \( \ell(2, 3) \). First, notice that \( t(K_s[2, 3]) \) is contained in \( B_2(X) \) entirely. Second, observe that the portion of the tube \( t(K \setminus K_s) \) contained in \( B_2(X) \) has a volume at least \((4\pi/3) + 6\pi \). It follows that \( \pi \ell(2, 3) < (4\pi/3)^3 - ((4\pi/3) + 6\pi) = 78\pi \), therefore \( \ell(2, 3) < 78 \). Let
\(a_1 = 78 - \ell[2,3]\). Observe that moving a segment of \(K_s\) closer to \(X\) will increase the integral on the right side of (3.1). So we can treat a part of \(K_s[3,4]\) that is of a total length of \(a_1\) as if it is part of \(K_s[2,3]\). If \(\ell[3,4] < a_1\), we will then take a part from \(K_s[4,5]\), and so on. In other word, we will force that \(\ell[2,3] = 78\) since this will only increase the integral (3.1). Having done so, we can then estimate \(\ell[3,4]\). Notice that now \(\ell[3,4]\) stands for the length of the leftover part of \(K_s[3,4]\) after part (or even all of it) is taken away to fill \(\ell[2,3]\) to 78. Notice that \(t(K_s[2,4])\) is contained in \(B_5(X)\) entirely and the portion of the tube \(t(K \setminus K_s)\) contained in \(B_5(X)\) has a volume at least \((4\pi/3) + 8\pi\). So we have

\[
\ell[2,4] = \ell[2,3] + \ell[3,4] < \frac{(4\pi)5^3}{3} - \frac{(4\pi)5}{3} + 8 = 157\frac{1}{3}.
\]

Since \(\ell[2,3] = 78\), we have \(\ell[3,4] < 4(5^3 - 4^3)/3 - 2 = 79\frac{1}{2}\). We will then force \(\ell[3,4] = 79\frac{1}{2}\) by moving portions of \(K_s[4,5]\) (or pieces that even further away if necessary) into \(B_3[3,4]\), as we did before. Similarly, assuming \(\ell[2,4] = 157\frac{1}{3}\), we will have \(\ell[4,5] < 4(6^3 - 5^3)/3 - 2 = 119\frac{1}{2}\). Continuing this argument, we force \(\ell[2,5] = 276\frac{2}{3}\), and we then get (since \(t(K \setminus K_s)\) is contained in \(B_7(X)\)) \(\ell[5,6] < 4(7^3 - 6^3)/3 - 2 = 168\frac{1}{2}\). After forcing \(\ell[2,6] = 457\frac{1}{3} - 4\pi\), we can then force \(\ell[n-1,n] = 4((n+1)^3 - n^3)/3\) for any \(n \geq 7\). Let \(B_s[N-1,N]\) be the last shell containing parts of \(K_s\) after the above procedure (called maximum packing of the tubes in \([\mathbf{BS}]\)), then we have

\[
L(K) - 4\pi = \sum_{k=3}^{N} \ell[k-1,k] \leq 4(N + 1)^3/3 - 4\pi.
\]

In other word, we will be able to pack \(K_s\) into \(B_n(X)\) for \(N = \lfloor(3L(K)/4)^{1/3}\rfloor\) where \([x]\) denotes the integer part of \(x\).

Apparently, if \(L(K)\) is relatively small, packing \(K_s\) into \(B_s[2,3]\) will not help us improving (1.5). However, if \(L(K)\) is not too small, a maximum packing of the tubes will improve (1.5).

Consider the case that \(78 + 4\pi < L(K) \leq 157\frac{1}{3} + 4\pi\), so that \(K_s\) can be packed into \(B_4(X)\) as discussed above. Notice that we have \(\ell[2,3] = \ell[2,2.5] + \ell[2.5,3]\) and \(\ell[2,2.5] < \frac{1}{2}(3.5^3 - 1) - 5 \approx 50.83\) (since the portion of \(t(K \setminus K_s)\) contained in \(B_{3,5}(\gamma(s))\) has volume at least \(5\pi + \frac{4\pi}{3}\)). It follows that \(\ell[2.5,3] \approx 78 - 50.83 = 27.17\). Therefore,

\[
\int_{K_s[2,3]} \frac{1}{\gamma(t) - \gamma(s)} dt \leq \frac{50.84}{4} + \frac{27.17}{2.5^2} < 17.06.
\]

Thus,

\[
\int_{K_s} \frac{1}{\gamma(t) - \gamma(s)} dt = \left(\int_{K_s[2,3]} + \int_{K_s[3,4]}\right) \frac{1}{\gamma(t) - \gamma(s)} dt < 17.06 + \ell[3,4]/9 = 17.06 + (L(K) - 78 - 4\pi)/9
\]

since \(\ell[3,4] = L(K) - 4\pi - \ell[2,3]\) in this case and \(\ell[2,3] = 78\) by our choice. This then leads to

\[
Cr(K) < \frac{1}{30\pi}L(K)(L(K) + 63), \quad 78 + 4\pi < L(K) \leq 157\frac{1}{3} + 4\pi.
\]
Notice that (3.7) is equivalent to

\[
L(K) > \left(-63 + \sqrt{63^2 + 144\pi Cr(K)}\right) / 2, 123 \leq Cr(K) \leq 349.
\]

The following table compares the lower bounds of the rope length by using (1.6) and (3.8). Notice that the improvement is larger as \(Cr(K)\) gets larger.

**Table 2**

<table>
<thead>
<tr>
<th>(Cr(K)) = 123</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
<th>349</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L(K) &gt; ) by (1.6)</td>
<td>87.77</td>
<td>95.93</td>
<td>109.3</td>
<td>121.1</td>
<td>131.7</td>
</tr>
<tr>
<td>(L(K) &gt; ) by (3.8)</td>
<td>90.57</td>
<td>102.5</td>
<td>122.1</td>
<td>139.5</td>
<td>155.3</td>
</tr>
</tbody>
</table>

Following the same approach, we can derive a few more such formulas for \(L(K)\) in different ranges, which we summarize in the following theorem. The details are left to our reader. The first inequality is (1.5).

**Theorem 3.1.** If \(K\) is a knot of unit thickness with length \(L(K)\) and crossing number \(Cr(K)\), then we have

\[
Cr(K) \leq \frac{1}{16\pi} L(K)(L(K) - 17.334)
\]

if \(24 \leq L(K) \leq 78 + 4\pi\),

\[
Cr(K) < \frac{1}{36\pi} L(K)(L(K) + 63)
\]

if \(78 + 4\pi < L(K) \leq 157\frac{1}{3} + 4\pi\),

\[
Cr(K) < \frac{1}{64\pi} L(K)(L(K) + 244.1)
\]

if \(157\frac{1}{3} + 4\pi < L(K) \leq 276\frac{2}{3} + 4\pi\), and

\[
Cr(K) < \frac{1}{100\pi} L(K)(L(K) + 544.1)
\]

if \(276\frac{2}{3} + 4\pi < L(K)\).

Notice that the above inequalities all work for \(L(K) \geq 24\) in fact. However, they yield the best upper bound in the specified range of \(L(K)\). Next, we wish to improve (1.1) for large values of \(L(K)\). Recall that for \(n \geq 7\), we are forcing \(\ell[n-1,n] = 4((n+1)^3-n^3)/3\) as long as \(L(K)\) is long enough so \(B_s[n-1,n]\) is not empty under the maximum packing of the tube \(t(K_S)\). It follows that

\[
\int_{K \times [n-1,n]} \frac{1}{\gamma(t) - \gamma(s)} \, dt \leq \ell[n-1,n] \left( \frac{(n+1)^3-n^3}{3} \right) = 4 + \frac{1}{n-1} + \frac{28}{3(n-1)^2}.
\]

Thus, if \(L(K) > 457\frac{1}{3}\) or equivalently \(N = [(3L(K)/4)^{1/3}] > 7\), we will have the following upper bound for the integral:

\[
\int_{K} \frac{1}{\gamma(t) - \gamma(s)} \, dt < 40.06 + 4(N-6) + 12\sum_{n=7}^{N} \frac{1}{n-1} + \frac{28}{3} \sum_{n=7}^{N} \frac{1}{(n-1)^2}.
\]

Combining this with \(\sum_{n=2}^{N} \frac{1}{n} < \ln N\) and \(\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\), we can further bound (3.14) by

\[
16.06 + 4N + 12(\ln(N-1) - 77/60) + 1.7 = 2.36 + 4N + 12\ln(N-1).
\]
Since \( N = [(3L(K)/4)^{1/3}] \), \( 12 \ln(N-1) < 12 \ln(N) = 4 \ln(3/4) + 4 \ln(L(K)) \). For \( L(K) > 1113 \), one can verify that 2.36+4\( N+12 \ln(N-1) < 1.21+4(3/4)^{1/3}L(K)+4 \ln(L(K)) < 3.751L(K)+4 \ln(L(K)) \). Thus we have

\[
Cr(K) < L(K) \left( 3.751L(K)+4 \ln(L(K)) \right) /4\pi
\]

(3.16)

if \( L(K) > 1113 \). On the other hand, for \( 24 < L(K) \leq 1113 \), one can verify that the following inequalities hold:

\[
\frac{L(K)+544.1}{100\pi} < \frac{3.751L(K)+4 \ln(L(K))}{4\pi}
\]

Therefore, (3.16) holds in general. Furthermore, we have

\[
\frac{3.751L(K)+4 \ln(L(K))}{4\pi} < \frac{6.46L(K)}{4\pi} \quad \text{if} \quad 1113 < L(K),
\]

(3.18)

\[
\frac{L(K)+544.1}{100\pi} < \frac{6.46L(K)}{4\pi} \quad \text{if} \quad 100 < L(K) \leq 1113,
\]

(3.19)

\[
\frac{L(K)-17.334}{16\pi} < \frac{6.46L(K)}{4\pi} \quad \text{if} \quad 24 < L(K) \leq 100.
\]

(3.20)

We can now summarize these into the following theorem.

**Theorem 3.2.** Let \( K \) be a knot of unit thickness, then

\[
Cr(K) < L(K) \left( 3.751L(K)+4 \ln(L(K)) \right) /4\pi
\]

(3.21)

and

\[
Cr(K) < \left( 6.46L(K) \right) /4\pi
\]

(3.22)

**Remark 3.3.** Observe that (3.22) improves the corresponding coefficient in [BS] from 11/4\( \pi \) to 6.46/4\( \pi \). Also notice that \( 3.751L(K) \) is an asymptotic upper bound of \( Cr(K) \) since \( \ln(L(K))/L(K) \to 0 \) as \( L(K) \to \infty \) and the coefficient of the \( L(K) \) term in (3.15) is actually less than 3.751/4\( \pi \) < 0.3. It is possible to improve (3.16) slightly for very large values of \( L(K) \). But this approach will not get the coefficient below \( (3/4)^{1/3} / \pi \), which is about 0.29. In [DE], it is shown that there exists a family of knots (with unit thickness) such that \( Cr(K) \geq 0.046L(K) \). So while further improvements of the constant 0.3 might be possible, the room of improvement can not exceed a factor of \( .29/.046 \approx 6.52 \). A graph of the various upper bounds in theorems 3.1 and 3.2 is shown below.

4. The Crossing Number of a Thick Link

This section is devoted to improve the result (1.3) obtained in [DER2] regarding (smooth) thick links in two aspects. First, we will improve the bound of the constant \( c \). Second, our result here holds for the crossing number between the two components, not just the linking number.
Let \((K_1, K_2)\) be a link of unit thickness with two components \(K_1\) and \(K_2\). Let \(L_1\) and \(L_2\) be the lengths of \(K_1\) and \(K_2\) respectively and let \(\gamma_1(t), \gamma_2(t)\) be the arc-length parameterized equations of \(K_1\) and \(K_2\) respectively. Recall that the average crossing number \(a(K_1, K_2)\) between \(K_1\) and \(K_2\) is given by (2.3), that is:

\[
a(K_1, K_2) = \frac{1}{4\pi} \left( \int_{K_1} \int_{K_2} + \int_{K_2} \int_{K_1} \right) \left| \frac{[\dot{\gamma}_1(t), \dot{\gamma}_2(s), \gamma_1(t) - \gamma_2(s)]}{|\gamma_1(t) - \gamma_2(s)|^3} \right| dtds.
\]

It follows that

\[
a(K_1, K_2) \leq \frac{1}{2\pi} \int_{K_1} \int_{K_2} \frac{1}{|\gamma_1(t) - \gamma_2(s)|^2} dsdt
\]

As we did in the last section, we need to bound the integral

\[I_s = \int_{K_2} \frac{1}{|\gamma_1(t) - \gamma_2(s)|^2} ds\]

for each \(t\). If \(L_1\) or \(L_2\) are less than \(4\pi\), \(K_1\) and \(K_2\) obviously are not linked hence the minimal crossing number between them would be zero. So we will assume that \(L_i \geq 4\pi\) for \(i = 1, 2\). Thus the arguments in Section 3 still hold if \(K_s\) is replaced by \(K_2\) except that \(K_1\) has no effect on the length of \(K_2\) so some small adjustments are needed. We state the following theorem without a proof. The details are left to our reader.

**Theorem 4.1.** If \((K_1, K_2)\) is a two-component link of unit thickness, then the crossing number \(Cr(K_1, K_2)\) between \(K_1\) and \(K_2\) is bounded above by

\[
\begin{align*}
(4.3) & \quad \frac{1}{8\pi} L_1 L_2 & \text{if } L_2 \leq 78, \\
(4.4) & \quad \frac{1}{18\pi} L_1 (L_2 + 75.6) & \text{if } 78 < L_2 \leq 157\frac{1}{3}, \\
(4.5) & \quad \frac{1}{32\pi} L_1 (L_2 + 256.7) & \text{if } 157\frac{1}{3} < L_2 \leq 276\frac{2}{3}, \\
(4.6) & \quad \frac{1}{50\pi} L_1 (L_2 + 556.7) & \text{if } 276\frac{2}{3} < L_2.
\end{align*}
\]
In general, \( Cr(K_1, K_2) \) is bounded above by

\[
L_1 \left( 3.751 L_2^\frac{1}{2} + 4 \ln L_2 \right) / 2\pi.
\]

The above holds if \( L_1 \) and \( L_2 \) are exchanged.

Asymptotically, this implies that the linking number \( Lk(K_1, K_2) \) between \( K_1 \) and \( K_2 \) is bounded by

\[
\min \left\{ \frac{3.751}{4\pi} L_1 L_2^\frac{1}{2}, \frac{3.751}{4\pi} L_2 L_1^\frac{1}{2} \right\}.
\]

if \( L_1 \) or \( L_2 \) is very large. Similar to the last section, we observe that

\[
3.751 L_1^\frac{1}{2} + 4 \ln L < 6.46 L_1^\frac{1}{2}
\]

if \( L > 1113 \),

\[
\frac{1}{25} (L + 556.7) < 6.46 L_1^\frac{1}{2}
\]

if \( 78 < L \leq 1113 \), and

\[
\frac{1}{4} L < 6.46 L_1^\frac{1}{2}
\]

if \( 4\pi \leq L \leq 78 \). Thus we have the following

**Theorem 4.2.** If \( (K_1, K_2) \) is a two-component link of unit thickness, then

\[
Cr(K_1, K_2) < \min \left\{ \frac{6.46}{2\pi} L_1 L_2^\frac{1}{2}, \frac{6.46}{2\pi} L_2 L_1^\frac{1}{2} \right\}.
\]

Consequently,

\[
Lk(K_1, K_2) < \min \left\{ \frac{1.62}{\pi} L_1 L_2^\frac{1}{2}, \frac{1.62}{\pi} L_2 L_1^\frac{1}{2} \right\}.
\]

The last inequality improves the constant \( c \) in [DER2] from \( 26/\pi \) to about \( 1.62/\pi \).

**References**


Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223

*E-mail address:* ydiao@uncc.edu

Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101

*E-mail address:* claus.ernst@wku.edu