A Generalization of Beatty's Theorem

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Abstract

In 1926 Sam Beatty made the following discovery, which he posed as a problem in [1]. If a is a positive irrational number, the sequences $m(1+a), m = 1, 2, \ldots$ and $n(1 + a^{-1}), n = 1, 2, \ldots$ together contain exactly one number from each of the intervals $(k, k+1), k = 1, 2, 3, \ldots$. The problem was solved by Ostrowski and Aitken[4] and generalized to a larger class of sequences by Lambeck and Moser[3]. The authors are grateful for UNC Charlotte undergraduate James Rudzinski for suggesting the combinatorial game that lead to this paper. The purpose of this paper is to extend this theorem from sequences to continuous functions. We first establish some notation. Let $P = (a_0 = 0, a_1, a_2, \ldots)$ be a strictly increasing unbounded sequence of real numbers. For any nonnegative number x, let $\lfloor x \rfloor_P$ be the largest member of P that does not exceed x, and for positive x,

$$\mathsf{x_P} = \begin{cases} \lfloor x \rfloor_P & \text{if } x \text{ is not in } P \\ a_{i-1} & \text{if } x \in P \text{ and } \lfloor x \rfloor_P = a_i. \end{cases}$$

Also, if $t > 0$, we define $N_t = \{0, t, 2t, 3t, 4t, \cdots\}$ and $N_t^+ = N_t \backslash \{0\}$.

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