Joint Ventures, Risk Sharing and Optimal Contract Design *

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Abstract

We analyze the problem of the optimal sharing of risks in a joint venture, when the two parties to the joint venture each have pre-existing risky projects and the new project has deterministic initial cost but random profit potential. The risks of the joint venture, as well as those of the pre-existing projects are driven by correlated Brownian motions and there is no debt or hidden information. We derive (i) the optimal equity investment stake per partner and the associated final ownership allocations and show that these are independent of the relative values of the ex-ante investments of the partners (ii) the conditions for mutual gain from risk sharing (iii) myopic, static and dynamic risk sharing rules. We show that the optimal stake by any partner is dependent on (a) a joint relative profitability index (time dependent) and a payoff function (time -invariant). The optimal risk sharing plan may or may not reduce overall enterprise risk depending on the correlation of new project cash flows with existing ones. The optimal sharing rule need not be linear and mutually beneficial risk sharing is possible even if the parties have the same degree of risk aversion. Finally we investigate the requirements for the partners to proceed with the development of the project rather than exercising an option to delay implementation.

1 Introduction

Many problems involve decision making in a principal agent type framework, or in a setting where multiple principals, of more or less equal standing, try to seek conditions for cooperation and for fruitful risk-sharing. If the parties face independent, identically distributed risks and share risk preference profiles, they can each gain by pooling their risks. The problem of sharing risks optimally for parties with different endowment levels, different risk profiles, different levels of incentives to cooperate is inherently more difficult to tackle.

Such problems arise in many areas such as joint ventures, syndication, and partnership agreements of all sorts. Moreover, the product and input markets in which these parties operate may be characterized by varying degrees of competition. Not only that, but there are typically adverse selection and moral hazard issues. All of these factors make the problem of determining an optimal level of risk sharing truly difficult.

Joint ventures create synergies by combining core skills of the partners to form a distinctive organizational structure with unique capabilities that neither partner could perhaps provide alone, or at least not as efficiently. However, there are many additional important sources of synergies: risk sharing, flexibility of operation, ability to screen projects better etc. Such a list of possibilities and scenarios is difficult to tackle in a universal sweep.

This paper does not try to address all of these issues. We deal with the problem of optimal risk sharing in a framework where we abstract from the product markets in which the joint venture products are sold and from moral hazard and most incentive issues. Since there are typically many factors at work behind a joint venture decision it is hard to attribute a particular motive as the unique/key driving force behind the decision.

The evidence on the value created by joint ventures is mixed. Joint ventures are shown to create value for the venture partners, as measured by the increased market value of the partner firms, McConnell and Nantes (1985). They find positive mean excess returns for joint venture partners but do not ascertain the factors behind such gains. Johnson and Houston (2000), distinguish between domestic horizontal and vertical joint ventures. They find that only horizontal joint ventures create synergistic gains that are shared by all partners. Vertical joint ventures only benefit suppliers. Moreover they assert not to have found any evidence of a risk sharing motive.[1]

On the other hand, Margitt, Broll and Mallick (1995) show that international joint ventures can dominate direct foreign investment, thus providing superior benefits for all partners. Margitt et al. attest to the benefits from a risk sharing plan. The joint venture solution provides for an increased level of overall investment relative to the FDI approach, and both parties are bet-

[1]We make no assumptions on the nature of the joint venture except that it is one of scale expansion of the previous activities of the venture partners. We on the other hand provide clear conditions where the risk sharing motive predominates and is coincident with individual expected utility maximization of each of the venture partners.

ter off. Amit, Mueller and Glosten (2000), examine the relationship between entrepreneurs and venture capitalists and the rationale behind the decision of entrepreneurs to share risk by bringing in outside participation. Their model is one of information asymmetry and moral hazard. The entrepreneurs decide that the benefits from risk sharing outweigh the agency costs of involving venture capitalists. The entrepreneur effectively turns over to the outside investors the right to determine the sharing arrangements. Brander, Amit and Antweller (2002) also examine joint investments in projects by venture capitalists and analyze risk sharing as one of many possible motives for syndication efforts. Chan et al. (1997) also find benefits from strategic alliances, which are a form of joint operation. Once the alliance is announced, positive mean price reactions are generated.

So it is relatively clear that there is agreement that some forms of joint ventures generate value for their partners and stake holders. What is less clear is what precisely are the motivations for entering into joint ventures, when both parties have the same risk assessment of the common project but differing degrees of risk aversion. How do we determine whether risk sharing is a key factor and what factors determine relative equity stakes. Equity investment stakes in joint ventures are determined by Nash bargaining in some papers e.g. Darrough and Stoughton (1989), (Lee 2004) and in other cases by means of option theoretic arguments, Metrick (2007) and Blenman and Clark (2005). Should a given equity stake lead to a similar allocation of cashflow? It is clear that the answer is a qualified no. In many cases, there is significant disparity between investment stakes and control rights. This is especially the case in international joint ventures, where it often occurs that the external investment stake is stipulated by law.

In this paper we attempt to address the question of how should the optimal investment stakes, and allocation shares be related when they are determined endogenously, as the parties focus on a joint venture as a means of sharing risk. We assume that there are no informational asymmetries and there is cooperation between the joint venture partners. We abstract from organizational structures as the parties are assumed to act to jointly improve their individual expected utility. We find conditions for the joint venture solution to dominate any solution that implies individual project choice by a particular partner acting in isolation.

We present a unified framework, where we can synthesize and explain the sharing rules of several papers in the literature. However, our results go beyond a mere generalization of previous results as we show that a key element in the determination of investment stakes and optimal sharing rules, hinges on the notion of a joint relative profitability index. The value of this index is the key to the generalization of our results. It is the direct outcome of the maximization of a preference weighted sum of the expected utilities of the partners. The paper is organized as follows. Section 1 is the introduction. Section 2 lays out the model and its assumptions and solves for the optimal risk sharing rule and optimal investment stake. The properties of the optimal solutions are analyzed. Section 3 investigates the optimal timing of a joint venture investment. Section 4 summarizes our results.

2 The Model and Its Solution

There are two investors who could potentially share the development cost and future profit of a joint venture. They jointly and individually have more than enough funds to finance the project on their own.[2] The project's cost is known at initial time as a positive constant c_v, and the profit is driven by a Brownian motion $B_v(t)$ with mean rate of return μ_v, volatility σ_v and initial value y_v, all of which are positive constants.[3] Therefore if the ownership of the joint venture is transferred or if the entity goes public at any given time t, the profit of the joint venture turns out to be $y_v e^{\mu_v t} + \sigma_v B_v(t)$. Suppose the risk-free rate is another positive constant $r < \mu_v$, then the excess profit will be $y_v e^{\mu_v t} - c_v e^{rt} + \sigma_v B_v(t)$.[4]

[2]This assumption does not rule out the possibility that the principal party to the joint venture agreement, if there is one, has enough funds to complete the project unilaterally, but wants to guard against either construction cost risks or profit flow risks. We abstract from the first possibility in our analysis, mainly as a simplifying assumption as the main results would not be affected by random construction costs.

[3]They openly share information about the prospect of the new project and its associated risk. They do not have to agree on a common risk assessment of existing projects that each party has upon entering the joint venture.

[4]The setup does not preclude the case where one partner initially identifies the opportunity, incurs some costs and then decides to take on a partner to share the project risk. In this case, c_v includes these preliminary costs. If there are such costs, it makes the ensuing analysis even stronger as there will be a greater incentive to participate in the project on a shared basis, as any setup costs will be borne solely by the entity that initially screened

In particular, the excess profit $y_v - c_v$ at initial time $t = 0$ should be negative to reflect the cost for immediate termination of the project, thus preventing the joint venture parties from entering and exiting the deal swiftly to make an arbitrage profit. The decision whether to enter the joint venture or not depends on the individual investor's pre-existing investment conditions and the characteristics of the new project[5]. Suppose investor one has an initial cash position c_1, and investment in a risky project with initial value y_1, mean rate of return μ_1 and volatility σ_1, all of which are assumed to be positive constants. Then her wealth process without participating in the joint venture will turn out to be

$$X_{10}(t) = c_1 e^{rt} + y_1 e^{\mu_1 t} + \sigma_1 B_1(t),$$

where Brownian motion $B_1(t)$ is assumed to be correlated with $B_v(t)$ where $dB_1(t)dB_v(t) = \rho_1 dt$, $-1 \leq \rho_1 \leq 1$. When the correlation satisfies $\rho_1 = 0$, these two risky projects are independent. Similarly, the parameters for the second investor are positive constants c_2, μ_2, σ_2 and y_2, and the wealth process is

$$X_{20}(t) = c_2 e^{rt} + y_2 e^{\mu_2 t} + \sigma_2 B_2(t),$$

the project.

[5]We explicitly assume that there are no side payments from any entity to the other, to induce participation in the new venture. In many international joint ventures such payments are also explicitly prohibited. Such payments could be incorporated within the confines of our setup, but the added complexity does not bring any additional deep insights and so we avoid this line of investigation at this stage. For more details, see the paper by Ross (1973) where the principal agent problem is analyzed. Our problem is in fact a principal/principal problem.

where $dB_2(t)dB_v(t) = \rho_2 dt$, $-1 \le \rho_2 \le 1$. The preference of each investor is governed by a CARA utility function with risk aversion parameters γ_1 and γ_2 respectively. Thus their expected utilities are as follows if we fix the time horizon at $t = T$:

$$u_1^0 = E[1 - e^{-\gamma_1 X_{10}(T)}] = 1 - e^{-\gamma_1(c_1 e^{rT} + y_1 e^{\mu_1 T}) + \frac{1}{2}\gamma_1^2\sigma_1^2 T}, \quad (1)$$

$$u_2^0 = E[1 - e^{-\gamma_2 X_{20}(T)}] = 1 - e^{-\gamma_2(c_2 e^{rT} + y_2 e^{\mu_2 T}) + \frac{1}{2}\gamma_2^2\sigma_2^2 T}. \quad (2)$$

To make sure that the investors are risk averse, we assume $\gamma_1 > 0$ and $\gamma_2 > 0$. The incentive for the investors to participate in the joint venture is to either improve their return or reduce their risk or both. In terms of the utility function, they would like to increase u_1^0 and u_2^0 respectively. Another incentive for joint ventures is that one party is unwilling to bear the total cost and risk of the project and seek a partner to share both objectives[6]. In case either one of them takes up the risky project alone, the individual wealth processes are

$$X_{11}(t) = c_1 e^{rt} + y_1 e^{\mu_1 t} + \sigma_1 B_1(t) - c_v e^{rt} + y_v e^{\mu_v t} + \sigma_v B_v(t),$$

$$X_{21}(t) = c_2 e^{rt} + y_2 e^{\mu_2 t} + \sigma_2 B_2(t) - c_v e^{rt} + y_v e^{\mu_v t} + \sigma_v B_v(t),$$

[6]As mentioned earlier, even if the investor has the funds to act independently, the risks associated with the revenue/profit flows may invoke second thoughts. Such types of behaviors are commonly seen in joint ventures that involve oil exploration and development, reservation systems, etc. They are even more pronounced when the projects are international in nature and the counter-party, partner is a sovereign or quasi-governmental organization.

and the corresponding expected utility functions can be calculated as

$$u_1^1 = E[1 - e^{-\gamma_1 X_{11}(T)}]$$
$$= 1 - e^{-\gamma_1((c_1-c_v)e^{rT}+y_1e^{\mu_1 T}+y_ve^{\mu_v T})+\frac{1}{2}\gamma_1^2(\sigma_1^2+2\rho_1\sigma_1\sigma_v+\sigma_v^2)T}, \quad (3)$$
$$u_2^1 = E[1 - e^{-\gamma_2 X_{21}(T)}]$$
$$= 1 - e^{-\gamma_2((c_2-c_v)e^{rT}+y_2e^{\mu_2 T}+y_ve^{\mu_v T})+\frac{1}{2}\gamma_2^2(\sigma_2^2+2\rho_2\sigma_2\sigma_v+\sigma_v^2)T}. \quad (4)$$

Now suppose investor one will bear αc_v of the initial cost of the joint venture, and investor two $(1-\alpha)c_v$, where $0 \le \alpha \le 1$. We define the *risk-sharing contract* to be a rule to calculate the percentage sharing of the equity stake of the joint venture depending on the percentage sharing of the initial cost:

- investor one receives profit $\lambda(\alpha)\,(y_ve^{\mu_v t} + \sigma_v B_v(t))$, and
- investor two receives profit $(1-\lambda(\alpha))\,(y_ve^{\mu_v t} + \sigma_v B_v(t))$.

Note that the proportion $\lambda(\alpha)$ of the equity stake of the joint venture $y_ve^{\mu_v t} + \sigma_v B_v(t)$ investor one owns at time t is a function of the original proportion of the cost sharing: α. For this risk-sharing contract to be *acceptable*, we impose the following conditions on the function $\lambda(\alpha)$:

(a) $\lambda(0) = 0, \lambda(1) = 1$, and

(b) $\lambda(\alpha)$ is an increasing and continuously differentiable function on $[0, 1]$.

If both investors enter the joint venture with risk-sharing contract λ, then their wealth process will become

$$\begin{aligned} X_{1v}(t) &= c_1 e^{rt} + y_1 e^{\mu_1 t} + \sigma_1 B_1(t) - \alpha c_v e^{rt} + \lambda(\alpha)\left(y_v e^{\mu_v t} + \sigma_v B_v(t)\right) \\ &= \beta_1(\alpha, t) + \Sigma_1(\alpha) W_1(t), \\ X_{2v}(t) &= c_2 e^{rt} + y_2 e^{\mu_2 t} + \sigma_2 B_2(t) - (1-\alpha) c_v e^{rt} + (1-\lambda(\alpha))\left(y_v e^{\mu_v t} + \sigma_v B_v(t)\right) \\ &= \beta_2(\alpha, t) + \Sigma_2(\alpha) W_2(t), \end{aligned}$$

where we have defined

$$\beta_1(\alpha, t) = (c_1 - \alpha c_v) e^{rt} + y_1 e^{\mu_1 t} + \lambda(\alpha) y_v e^{\mu_v t}, \tag{5}$$

$$\beta_2(\alpha, t) = (c_2 - (1-\alpha) c_v) e^{rt} + y_2 e^{\mu_2 t} + (1-\lambda(\alpha)) y_v e^{\mu_v t}, \tag{6}$$

$$\Sigma_1(\alpha) = \sqrt{\sigma_1^2 + 2\lambda(\alpha)\rho_1\sigma_1\sigma_v + \lambda(\alpha)^2\sigma_v^2}, \tag{7}$$

$$\Sigma_2(\alpha) = \sqrt{\sigma_2^2 + 2(1-\lambda(\alpha))\rho_2\sigma_2\sigma_v + (1-\lambda(\alpha))^2\sigma_v^2}, \tag{8}$$

$$W_1(t) = \frac{\sigma_1}{\Sigma_1(\alpha)} B_1(t) + \frac{\lambda(\alpha)\sigma_v}{\Sigma_1(\alpha)} B_v(t), \tag{9}$$

$$W_2(t) = \frac{\sigma_2}{\Sigma_2(\alpha)} B_2(t) + \frac{(1-\lambda(\alpha))\sigma_v}{\Sigma_2(\alpha)} B_v(t). \tag{10}$$

By Lévy's Theorem, we know that $W_1(t)$ and $W_2(t)$ are both Brownian motions. Comparing the wealth process for the first investor $X_{10}(t)$, where there were only pre-existing investment, to $X_{1v}(t)$, where the joint venture risk-sharing plan is added, we see that its mean changed from $c_1 e^{rt} + y_1 e^{\mu_1 t}$ to $\beta_1(\alpha, t)$, and the variance changed from σ_1 to $\Sigma_1(\alpha)$. The updated expected utilities are

$$u_1(\alpha) = E[1 - e^{-\gamma_1 X_{1v}(T)}] = 1 - e^{-\gamma_1 \beta_1(\alpha,T) + \frac{1}{2}\gamma_1^2 \Sigma_1(\alpha)^2 T}, \tag{11}$$

$$u_2(\alpha) = E[1 - e^{-\gamma_2 X_{2v}(T)}] = 1 - e^{-\gamma_2 \beta_2(\alpha,T) + \frac{1}{2}\gamma_2^2 \Sigma_2(\alpha)^2 T}. \tag{12}$$

In case a fixed point for the risk sharing contract exists, such that $\lambda(\tilde{\alpha}) = \tilde{\alpha}$, the proportion of cost-sharing and risk-sharing are equal. However, the negotiation between the two investors might not settle on a mutually agreed proportion $\tilde{\alpha}$ if either one or both has the potential to gain on the expected utility at other values of α. Therefore, our task is to find a well-designed acceptable risk-sharing contract λ, such that we can find an *efficient risk-sharing plan* $(\alpha^*, \lambda(\alpha^*))$ where

$$u_1'(\alpha^*) = 0, \quad u_2'(\alpha^*) = 0; \tag{13}$$

$$u_1'(\alpha) > 0, \quad u_2'(\alpha) > 0, \quad \forall \alpha \in (0, \alpha^*); \tag{14}$$

$$u_1'(\alpha) < 0, \quad u_2'(\alpha) < 0, \quad \forall \alpha \in (\alpha^*, 1). \tag{15}$$

Note that this set of sufficient optimality conditions is very strong. It implies that

$$u_1(\alpha^*) = \max_{\alpha \in [0,1]} u_1(\alpha), \quad u_2(\alpha^*) = \max_{\alpha \in [0,1]} u_2(\alpha).$$

At the optimal plan α^*, both investors' expected utilities are simultaneously

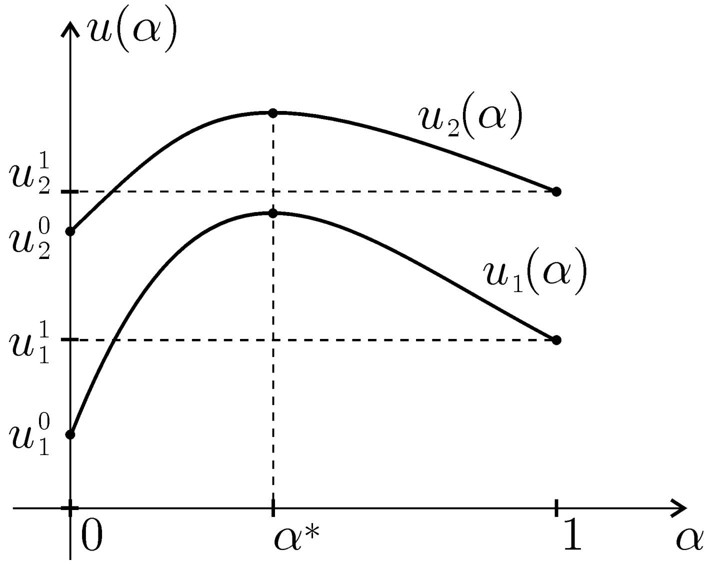

Figure 1: The expected utilities functions achieve maximums at the same α^* for both investors.

maximized from equation from (13), (14) and (15). We observe that $u_1^0 = u_1(0)$ and $u_1^1 = u_1(1)$ and at optimal risk-sharing plan α^*, investor one's expected utility is higher than what he would obtain if he takes the risky project alone or not participating in it at all:

$$u_1(\alpha^*) > u_1^0, \quad u_1(\alpha^*) > u_1^1. \tag{16}$$

The same holds true for investor two as well:

$$u_2(\alpha^*) > u_2^0, \quad u_2(\alpha^*) > u_2^1, \tag{17}$$

where we denote $u_2^0 = u_2(0)$ and $u_2^1 = u_2(1)$. Therefore, no one has any incentive to move away from this optimizing result. Conditions (13)-(17) are summarized in Figure 1. The above optimality conditions use only the first derivatives. If the expected utility functions are concave and twice differentiable with respect to the variable α, then conditions

$$u_1^{'}(\alpha^*) = 0, \quad u_2^{'}(\alpha^*) = 0; \tag{18}$$

$$u_1^{''}(\alpha) < 0, \quad u_2^{''}(\alpha) < 0, \quad \forall \alpha \in (0,1). \tag{19}$$

are sufficient for α^* to be an efficient risk-sharing plan. In both cases, the optimal risk-sharing plan will turn out to be unique.

Normally speaking, for a specific risk sharing contract $\lambda(\alpha)$, we do not expect the expected utility will be maximized at the same point α^*. Mathematically, the two equations in (13), namely $u_1'(\alpha) = 0, u_2'(\alpha) = 0$ usually have no solution if there is only one free variable α. To obtain such strong consensus with a solution α^*, we need to introduce another free variable. In this paper, we choose risk sharing contracts with a free parameter, say b. Then the derivatives of the expected utilities u_1' and u_2' will depend on both α and b, and it is often the case that a pair of solution (b^*, α^*) exists. The implication is that there is a risk-sharing contract (characterized by b^*), which allows both investors to arrive at the same risk-sharing plan as their optimal. In this Principle-Principle (vs. Principle-Agent) setting,

the design of the risk-sharing contracts is critical for the strength of the optimality condition and consequently the robustness of the contracts from re-negotiation incentive. The main results will be stated in Theorem 2.1 and Theorem 2.2, and we will demonstrate some contract design in the examples that follow.

A weaker condition will be a *Pareto efficient risk-sharing plan* where it is not possible to move from an allocation $\bar{\alpha}$ such that one investor is better off and the other will not be worse off, for example,

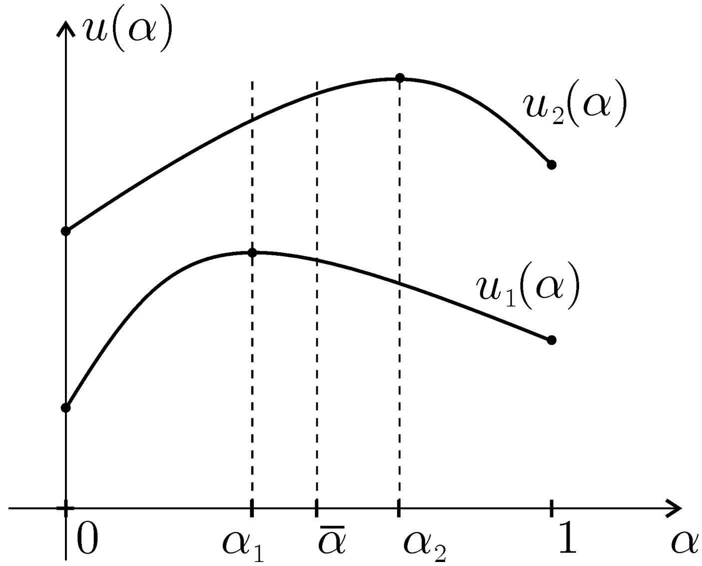

Figure 2: The expected utilities functions achieve their maximum values at different α's for the two investors. It is a much stronger condition if we require $\alpha_1 = \alpha_2 = \alpha^*$.

$$u_1^{'}(\bar{\alpha}) < 0, \quad u_2^{'}(\bar{\alpha}) > 0; \tag{20}$$

$$u_1^{'}(\alpha) > 0, \quad \forall \alpha \in (0, \alpha_1), \quad u_1^{'}(\alpha) < 0, \quad \forall \alpha \in (\alpha_1, 1); \tag{21}$$

$$u_2^{'}(\alpha) > 0, \quad \forall \alpha \in (0, \alpha_2), \quad u_2^{'}(\alpha) < 0, \quad \forall \alpha \in (\alpha_2, 1), \tag{22}$$

where $\alpha_1 < \bar{\alpha} < \alpha_2$. The Pareto efficient conditions is also the characterization of the optimal solution to the problem where investors cooperate to maximize weighted sum of utilities

$$\max_{\alpha} u_1(\alpha) + \bar{w}\, u_2(\alpha), \tag{23}$$

and the exact weight $\bar{w}$ corresponds to the particular choice of $\bar{\alpha}$. **The scalarization (23) of the vector optimization problem (the maximum of a vector composed of $u_1(\alpha)$ and $u_2(\alpha)$) that yields the same Pareto optimal solutions is essentially the duality approach in optimization techniques. It is well-explained in Sections 2.6.3 and 4.7.5 in Boyd and Vandenberghe (2004).** We will not study the risk-sharing plans that give Pareto efficiency in this paper as we focus on finding contracts, that are immune to re-negotiation, and which satisfy the stronger optimality conditions given in (13)-(15) or (18)-(19). The difference can be seen by comparing Figure 2 to Figure 1. However, it is obvious that the optimal solution to our problem in search of efficient risk-sharing contract, if exists, is also Pareto efficient characterized by conditions (20)-(22) for any $\bar{\alpha}$. Therefore, they are also the solution to the problem set up as in

Ross (1973) and Bolton and Dewatripont (2005) with any weight $\bar{w}$.[7]

Let us define two auxiliary functions

$$f_1(\alpha, T) = c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v + \gamma_1 T \sigma_v^2 \lambda(\alpha)\right) \lambda'(\alpha),$$
$$f_2(\alpha, T) = c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v + \gamma_2 T \sigma_v^2 (1 - \lambda(\alpha))\right) \lambda'(\alpha).$$

Theorem 2.1. *Suppose $\lambda(\alpha)$ is an acceptable risk-sharing contract. An efficient risk-sharing plan $(\alpha^*, \lambda(\alpha^*))$ exists if and only if for $\alpha^* \in (0,1)$ the following conditions hold:*

$$f_1(\alpha^*, T) = 0, \quad f_2(\alpha^*, T) = 0; \tag{24}$$
$$f_1(\alpha, T) < 0, \quad f_2(\alpha, T) > 0, \quad \forall \alpha \in (0, \alpha^*); \tag{25}$$
$$f_1(\alpha, T) > 0, \quad f_2(\alpha, T) < 0, \quad \forall \alpha \in (\alpha^*, 1). \tag{26}$$

Remark 2.1. *The meaning of the optimality conditions are as follows:*

- *(24) ⇔ (13): α^* is a critical point for both expected utilities $u_1(\alpha)$ and $u_2(\alpha)$.*
- *(25) and (26) ⇔ (14) and (15): $u_1(\alpha)$ and $u_2(\alpha)$ are strictly increasing on the interval $(0, \alpha^*)$ and decreasing on $(\alpha^*, 1)$, therefore the critical point α^* is the global maximum point for both.*

[7] Ross (1973), Bolton and Dewatripont (2005), Cvitanic et al. (2008) whose approaches are closest in spirit to us work in a framework of an agent and a well defined principal. On the other hand ours is world of symmetry where there are effectively two principals who act in extreme cooperation.

If we would like to impose a stronger condition where the expected utility functions are concave and twice differentiable, then the second order conditions in (19) can be stated as follows:

$$f_1(\alpha,T)^2 + \tfrac{1}{\gamma_1}\left(-y_v e^{\mu_v T} + \gamma_1 T\rho_1\sigma_1\sigma_v + \gamma_1 T\sigma_v^2\lambda(\alpha)\right)\lambda''(\alpha) + T\sigma_v^2(\lambda'(\alpha))^2 > 0, \quad (27)$$

$$f_2(\alpha,T)^2 - \tfrac{1}{\gamma_2}\left(-y_v e^{\mu_v T} + \gamma_2 T\rho_2\sigma_2\sigma_v + \gamma_2 T\sigma_v^2(1-\lambda(\alpha))\right)\lambda''(\alpha) + T\sigma_v^2(\lambda'(\alpha))^2 > 0. \quad (28)$$

Note that, as is often the case with exponential utility, the cash positions c_1 and c_2 do not affect the solutions. The mean rate of returns μ_1, μ_2 of the pre-existing portfolios also do not matter. Only the variances σ_1, σ_2 and the correlations ρ_1 and ρ_2 of the pre-existing projects affect the optimality conditions.

Proof. The partial derivatives of equations (5)-(8) with respect to the parameter α can be computed as

$$\begin{aligned}
\frac{\partial\beta_1(\alpha,t)}{\partial\alpha} &= -c_v e^{rt} + y_v e^{\mu_v t}\lambda'(\alpha) = -\frac{\partial\beta_2(\alpha,t)}{\partial\alpha},\\
\Sigma_1'(\alpha) &= \frac{\rho_1\sigma_1\sigma_v\lambda'(\alpha) + \sigma_v^2\lambda(\alpha)\lambda'(\alpha)}{\Sigma_1(\alpha)},\\
\Sigma_2'(\alpha) &= -\frac{\rho_2\sigma_2\sigma_v\lambda'(\alpha) + \sigma_v^2(1-\lambda(\alpha))\lambda'(\alpha)}{\Sigma_2(\alpha)}.
\end{aligned}$$

Then the derivatives of the expected utilities (11) and (12) are

$$u_1'(\alpha) = (u_1(\alpha) - 1)\left(-\gamma_1 \tfrac{\partial \beta_1(\alpha,T)}{\partial \alpha} + \gamma_1^2 T \Sigma_1(\alpha)\Sigma_1'(\alpha)\right)$$
$$= (u_1(\alpha) - 1)\gamma_1 f_1(\alpha, T),$$
$$u_2'(\alpha) = (u_2(\alpha) - 1)\left(-\gamma_2 \tfrac{\partial \beta_2(\alpha,T)}{\partial \alpha} + \gamma_2^2 T \Sigma_2(\alpha)\Sigma_2'(\alpha)\right)$$
$$= -(u_2(\alpha) - 1)\gamma_2 f_2(\alpha, T).$$

Since we have assumed $\gamma_1 > 0$ and $\gamma_2 > 0$, and we know that $u_1(\alpha) < 1$ and $u_2(\alpha) < 1$, we conclude (13) is equivalent to (24). In addition (14) is equivalent to (25); and (15) is equivalent to (26). Now we compute the second derivatives

$$u_1''(\alpha) = (u_1(\alpha) - 1)\gamma_1^2 \Big[f_1(\alpha, T)^2 + \tfrac{1}{\gamma_1}\left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v + \gamma_1 T \sigma_v^2 \lambda(\alpha)\right)\lambda''(\alpha) + T\sigma_v^2 (\lambda'(\alpha))^2\Big],$$
$$u_2''(\alpha) = (u_2(\alpha) - 1)\gamma_2^2 \Big[f_2(\alpha, T)^2 - \tfrac{1}{\gamma_2}\left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v + \gamma_2 T \sigma_v^2 (1 - \lambda(\alpha))\right)\lambda''(\alpha) + T\sigma_v^2 (\lambda'(\alpha))^2\Big],$$

and it is straight forward to find the second order conditions (27) and (28) which correspond to (19). ◇

Theorem 2.1 provides sufficient and necessary conditions for a function $\lambda(\alpha)$ to be a good risk-sharing contract which produces a definite nonnegotiable result for the joint venture participation at proportions $(\alpha^*, \lambda(\alpha^*))$. The risk sharing contracts $\lambda(\alpha)$ which satisfy these conditions are not unique, so are the optimal risk-sharing plans α^*. We expect the optimal sharing of the equity stake to vary according to the risk-sharing contracts. Thus the next theorem come as a total surprise which states affirmatively that the optimal equity stake sharing is an invariant. Define parameters

$$\beta = \frac{\gamma_2(\rho_2\sigma_2 + \sigma_v) - \gamma_1\rho_1\sigma_1}{(\gamma_1 + \gamma_2)\sigma_v} = \frac{\gamma_2}{\gamma_1 + \gamma_2}\left(\frac{\rho_2\sigma_2}{\sigma_v} + 1\right) - \frac{\gamma_1}{\gamma_1 + \gamma_2}\frac{\rho_1\sigma_1}{\sigma_v}, \quad (29)$$

$$\zeta = \frac{c_v e^{rT}}{y_v e^{\mu_v T} - \frac{\gamma_1\gamma_2 T}{\gamma_1+\gamma_2}(\rho_1\sigma_1\sigma_v + \rho_2\sigma_2\sigma_v + \sigma_v^2)}. \quad (30)$$

Theorem 2.2. *Suppose $\lambda(\alpha)$ is an acceptable risk-sharing contract, and $(\alpha^*, \lambda(\alpha^*))$ is the corresponding efficient risk-sharing plan. Then*

$$\lambda(\alpha^*) = \beta, \quad \lambda'(\alpha^*) = \zeta. \quad (31)$$

Remark 2.2. *Note that the optimal cost sharing can be calculated from either equations*

$$\alpha^* = \lambda^{-1}(\beta), \quad \alpha^* = (\lambda')^{-1}(\zeta). \quad (32)$$

It is necessary for the optimality conditions that the parameters satisfy

$$0 < \beta < 1, \quad \zeta > 0.$$

When $\beta = 0$, or equivalently, $\frac{\gamma_1}{\gamma_2} = \frac{\rho_2\sigma_2+\sigma_v}{\rho_1\sigma_1}$; or when $\beta = 1$, or equivalently,

$\frac{\gamma_1}{\gamma_2} = \frac{\rho_2\sigma_2}{\rho_1\sigma_1+\sigma_v}$, there is no risk-sharing agreement. The condition $0 < \beta < 1$ is equivalent to $0 < \gamma_2(\rho_2\sigma_2 + \sigma_v) - \gamma_1\rho_1\sigma_1 < (\gamma_1 + \gamma_2)\sigma_v$. Similarly, for the optimal risk-sharing plan to exist, ζ must be positive, or equivalently, $y_v e^{\mu_v T} > \frac{\gamma_1\gamma_2 T}{\gamma_1+\gamma_2}(\rho_1\sigma_1\sigma_v + \rho_2\sigma_2\sigma_v + \sigma_v^2)$.

PROOF. Condition (24) implies that

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T\rho_1\sigma_1\sigma_v + \gamma_1 T\sigma_v^2\lambda(\alpha^*)\right)\lambda'(\alpha^*) = 0, \qquad (33)$$

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T\rho_2\sigma_2\sigma_v + \gamma_2 T\sigma_v^2(1 - \lambda(\alpha^*))\right)\lambda'(\alpha^*) = 0. \qquad (34)$$

It is straight forward to check that the solutions to the above equations are exactly what we have in (31). ◇

An important discovery in the above theorem is that the optimal equity stake sharing of the joint venture $\lambda(\alpha^*)$ is a constant β for any risk sharing contract λ. Therefore this solution is very robust to any negotiation process through the selection of risk-sharing contracts. Another important feature of β is that it is time-invariant. This implies that after the risk-sharing contract has been signed, and cost sharing being carried out, neither party will have incentive for dynamic re-negotiation of their respective equity stake.

Regarding the optimal cost sharing α^*, we will see in the discussion after Example 2.2 that under fairly general conditions, $\alpha^* = g(\zeta, \beta)$ where the functional form of g is determined by the nature of the sharing rules λ employed. To get a better feel of what is implied by the linkage between optimal investment stake α^* and the optimal sharing of the profit, $\lambda(\alpha^*) = \beta$, define the *joint relative profitability index* ϕ *as* the reciprocal of the parameter ζ:

$$\phi = \frac{y_v e^{\mu_v T} - \frac{\gamma_1\gamma_2 T}{\gamma_1+\gamma_2}(\rho_1\sigma_1\sigma_v + \rho_2\sigma_2\sigma_v + \sigma_v^2)}{c_v e^{rT}}.$$

We observe that

$$\rho_1\sigma_1\sigma_v + \rho_2\sigma_2\sigma_v + \sigma_v^2 = \frac{(\Sigma_1(1))^2 - \sigma_1^2 + (\Sigma_2(0))^2 - \sigma_2^2}{2},$$

where $(\Sigma_1(1))^2 - \sigma_1^2$ is the excess unit-time risk if investor one takes up the joint venture alone, while $(\Sigma_2(0))^2 - \sigma_2^2$ is the excess unit-time risk if investor two takes up the joint venture alone. Therefore,

$$\begin{aligned} y_v e^{\mu_v T} - & \frac{\gamma_1\gamma_2 T}{\gamma_1+\gamma_2}(\rho_1\sigma_1\sigma_v + \rho_2\sigma_2\sigma_v + \sigma_v^2) \\ & = y_v e^{\mu_v T} - \frac{1}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}} \frac{(\Sigma_1(1))^2 - \sigma_1^2 + (\Sigma_2(0))^2 - \sigma_2^2}{2} T \end{aligned}$$

can be interpreted as joint risk-adjusted cash flow over period T, and thus we arrive to the name of joint relative profitability index. The existence of the optimal solution implies that $\zeta > 0$ and consequently $\phi > 0$, which in turn implies strong rationality on the part of the venture partners who need $\phi > 0$ for

$$y_v e^{\mu_v T} - \frac{\gamma_1\gamma_2 T}{\gamma_1+\gamma_2}(\rho_1\sigma_1\sigma_v + \rho_2\sigma_2\sigma_v + \sigma_v^2) = \phi c_v e^{rT},$$

or else they will not proceed with the project. The risk adjusted cash flows from the project is a weighted average of the sum of the extra variability in wealth, that would be triggered if each partner tried to do the project on their own. The weight is a simple average adjusted by risk aversion parameters $\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} = \frac{1}{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}}$. Since ϕ is time dependent, as well as the optimality conditions for function $\lambda(\alpha)$ in (24)-(26), there are timing issues associated with the best time to start the project and we will deal with these in a later section of the paper. We have thus shown that the optimal equity stake has two components, one that is time dependent, ϕ, and the other that is time-invariant, β.

In retrospect, when we are dealing with general utility functions, the main results from Theorem 2.1 and Theorem 2.2 will remain, although they might not be stated in an explicit form that is easy for calculation. The first order optimality conditions (24)-(26) will likely become implicit functions; and the optimal equity stake β in (31) will remain an invariant, but likely will have to be calculated numerically; however, the functional form g for calculating optimal cost sharing $\alpha^* = g(\zeta, \beta)$ will remain the same.

Next we give specific examples of risk sharing rules of the exponential and power kind. We will show that these different rules all affect the specific nature of the g function linking, α^* to ζ and β, i.e., $\alpha^* = g(\zeta, \beta)$. We start with an exponential sharing rule.

Example 2.1 (Exponential Sharing Rule)**.** *Suppose the following conditions* *hold*[8]*:*

$$0 < \beta < 1, \qquad \zeta > 0.$$

If $a \in (-\infty, 0) \cup (1, \infty)$ *is a solution (not necessarily unique) to the equation*

$$\frac{\zeta}{a-\beta} = \ln(\frac{a}{a-1}), \tag{35}$$

then we can define

$$b = \ln\left(\frac{a}{a-1}\right) \quad \textit{and} \quad \alpha^* = -\frac{1}{b}\ln\frac{\zeta}{ab}. \tag{36}$$

If in addition the following inequalities hold

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v\right) ab \;<\; 0, \tag{37}$$

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v + \gamma_2 T \sigma_v^2\right) ab \;>\; 0, \tag{38}$$

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v + \gamma_1 T \sigma_v^2\right) b(a-1) \;>\; 0, \tag{39}$$

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v\right) b(a-1) \;<\; 0. \tag{40}$$

Then we have an exponential acceptable risk-sharing contract

$$\lambda(\alpha) = -ae^{-b\alpha} + a,$$

and both investors' expected utilities are simultaneously maximized at the

[8]Note that both conditions are guaranteed by the existence of true risk sharing and strong rationality in the investment decision making process.

efficient risk-sharing plan (α^*, β).

Proof. Obviously $\lambda(0) = 0$. To get $\lambda(1) = 1$, we need

$$a(1 - e^{-b}) = 1 \quad \Leftrightarrow \quad b = \ln(\tfrac{a}{a-1}).$$

Note that we have either $b > 0, a > 1$ or $b < 0, a < 0$. Differentiating $\lambda(\alpha)$, we get

$$\lambda'(\alpha) = abe^{-b\alpha} = b(a - \lambda(\alpha)) > 0.$$

Note that $\lambda(\alpha)$ is an increasing function. We know from Theorem 2.2,

$$\lambda(\alpha^*) = \beta = -ae^{-b\alpha^*} + a, \quad \lambda'(\alpha^*) = \zeta = abe^{-b\alpha^*}.$$

To solve the above equations, it is natural to require that $0 < \beta < 1$ and $\zeta > 0$. Since $b = \ln(\frac{a}{a-1})$, we need to prove that there exist some solution 'a' to the equation

$$\tfrac{\zeta}{a-\beta} = \ln\left(\tfrac{a}{a-1}\right). \tag{41}$$

Then we will get a unique

$$\alpha^* = -\tfrac{1}{b}\ln\tfrac{\zeta}{ab},$$

given fixed a, b, ζ. Note that the assumption $0 < \beta < 1$ and the the fact that $\lambda(\alpha)$ increase from 0 to 1 on $\alpha \in [0,1]$, guarantees that $0 < \alpha^* < 1$.

For the case $\alpha \in (0, \alpha^*)$, we only need to show the inequality (25) at $\alpha = 0$ because the uniqueness of the critical point implies the signs of the first derivatives of the expected utilities will remain the same on the whole interval. Similarly, on $\alpha \in (\alpha^*, 1)$, we only need to show the inequality (26) at $\alpha = 1$. We need to verify

$$\begin{aligned} c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v + \gamma_1 T \sigma_v^2 \lambda(0)\right) \lambda'(0) < 0, \\ c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v + \gamma_2 T \sigma_v^2 (1 - \lambda(0))\right) \lambda'(0) > 0, \\ c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v + \gamma_1 T \sigma_v^2 \lambda(1)\right) \lambda'(1) > 0, \\ c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v + \gamma_2 T \sigma_v^2 (1 - \lambda(1))\right) \lambda'(1) < 0. \end{aligned}$$

With $\lambda(0) = 0$, $\lambda(1) = 1$, $\lambda'(0) = ab$, and $\lambda'(1) = b(a-1)$, we get the equivalent conditions (37) through (40). ◇

Similar to Example 2.1, we will provide the sufficient conditions on the parameters where efficient risk-sharing plans can be found for contracts which are fractional linear. Note that when $b = 1$ and $\zeta = 1$ in the following example, we have the affine sharing rule $\lambda(\alpha) = \alpha$, while the optimal sharing plan is still uniquely determined at $\alpha^* = \beta$.

Example 2.2 (Fractional Linear Sharing Rule)**.** *Define* $b = \frac{\zeta - \beta(1-\beta)}{(1-\beta)^2}$. *Assume the following conditions hold:*

$$0 < \beta < 1, \quad b > \frac{2\beta}{1+\beta}, \quad \zeta > 0.$$

If in addition the following inequalities hold

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v\right) b \quad < \quad 0, \tag{42}$$

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v + \gamma_2 T \sigma_v^2\right) b \quad > \quad 0, \tag{43}$$

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_1 T \rho_1 \sigma_1 \sigma_v + \gamma_1 T \sigma_v^2\right) \quad > \quad 0, \tag{44}$$

$$c_v e^{rT} + \left(-y_v e^{\mu_v T} + \gamma_2 T \rho_2 \sigma_2 \sigma_v\right) \quad < \quad 0. \tag{45}$$

Then we have an acceptable risk-sharing contract of fractional linear type

$$\lambda(\alpha) = \frac{b\alpha}{1 + \alpha(b-1)},$$

and both investors' expected utilities are simultaneously maximized at the efficient risk-sharing plan (α^*, β) *where*

$$\alpha^* = \frac{\beta}{b - (1-b)\beta} = \frac{\beta(1-\beta)}{\zeta}.$$

Many authors, in settings of moral hazard and adverse selection, have reported various other types of sharing rules, given an optimal investment stake. These include, square root type rule $\lambda(\alpha) = \sqrt{\alpha}$, linear or (affine) rule $\lambda(\alpha) = \alpha$, and power rule $\lambda(\alpha) = a\alpha^2 + (1-a)\alpha + \eta$, where η is a sum of higher order powers of α. Similarly, we can show, that there are other candidate sharing rules, for example, quadratic rules $\lambda(\alpha) = -(\alpha-1)^2 + 1; \lambda(\alpha) = -2(\alpha-1)^2 - \alpha + 2$. There are many such possible rules. Regardless of the choice of the functional form of these rules, the optimal investment stake and the payoff are uniquely defined in Theorem 2.1 and Theorem 2.2 in the context of this paper. The rule in place is unique given the characteristics of the project, the risk aversion parameters of the partners. We summarize a few of them in the following table where we have derived the optimal risk sharing plan from equation (32).

Acceptable Risk Sharing Contract	Optimal Cost Sharing	Optimal Equity Stake
$\lambda(\alpha) = \sqrt{\alpha}$	$\alpha^* = \beta^2 = \left(\frac{1}{2\zeta}\right)^2$	$\lambda(\alpha^*) = \beta$
$\lambda(\alpha) = -(\alpha-1)^2 + 1$	$\alpha^* = \sqrt{1-\beta} + 1 = 1 - \frac{\zeta}{2}$	$\lambda(\alpha^*) = \beta$
$\lambda(\alpha) = -ae^{-b\alpha} + a$	$\alpha^* = \frac{1}{b}\ln\frac{a}{a-\beta} = -\frac{1}{b}\ln\frac{\zeta}{ab}$	$\lambda(\alpha^*) = \beta$
$\lambda(\alpha) = \frac{b\alpha}{1+\alpha(b-1)}$	$\alpha^* = \frac{\beta}{b-(1-b)\beta} = \frac{\sqrt{\frac{b}{\zeta}}-1}{b-1}$	$\lambda(\alpha^*) = \beta$

Table 1: Summary of Sharing Rules and Optimal Investment Stakes

Note that it is very rare for the square root rule to produce an optimal risk sharing plan because the equality $\beta^2 = \left(\frac{1}{2\zeta}\right)^2$ normally will not hold. This is a consequence of the high requirements of the optimality condition imposed to have both equalities in (13) at the same α^*. However, in the exponential sharing rule we have one free parameter (not two because a and b is related by $b = \ln\left(\frac{a}{a-1}\right)$), and it is the same case in the fractional linear rule (parameter b). It was not difficult for us to find a pair of b and α^* where finally the equalities in (13) are satisfied. In the fractional linear case, this turn out to be straightforward

$$b = \frac{\zeta - \beta(1-\beta)}{(1-\beta)^2}, \quad \alpha^* = \frac{\beta(1-\beta)}{\zeta}.$$

The intuition is that, when there are infinitely many risk-sharing contracts parameterized by b, the Parato optimal risk sharing $\bar{\alpha}$ shown in Figure 2 normally exists for each contract. However, if we choose the best contract associated to a particular value b, then under easily calculable conditions (25) and (26) given in Theorem 2.1 both investors will achieve maximum expected utility at the same risk sharing plan $(\alpha^*, \lambda(\alpha^*))$.

Now we discuss the trade-off between return and risk by participation in the joint venture.

Theorem 2.3. *Given that $\lambda(\alpha)$ is an acceptable risk-sharing contract and $(\alpha^*, \lambda(\alpha^*))$ is the corresponding efficient risk-sharing plan. Then the risk adjusted contribution from participation in the project must be greater than $0.5\gamma_i$, for each partner, if the extra enterprise risk generated by the project is positive.*

Proof. For acceptable risk sharing to be efficient, it is necessary that $u_1(\alpha^*) > u_1^0$, and $u_1(\alpha^*) > u_1^1$. That is the expected utility for entity one, with sharing exceeds that from either going solo on the new project or abandoning it completely. Similarly, the expected utility for entity two is greater when there is shared participation in the project, instead of shouldering all its risks alone, i.e., $u_2(\alpha^*) > u_2^0$ and $u_2(\alpha^*) > u_2^1$. This and equations (1)-(4) and (11)-(12) imply for the first entity that

$$1 - e^{-\gamma_1\beta_1(\alpha^*,T)+\frac{1}{2}\gamma_1^2\Sigma_1(\alpha^*)^2T} > 1 - e^{-\gamma_1\beta_1(0,T)+\frac{1}{2}\gamma_1^2\Sigma_1(0)^2T}, \tag{46}$$

$$1 - e^{-\gamma_1\beta_1(\alpha^*,T)+\frac{1}{2}\gamma_1^2\Sigma_1(\alpha^*)^2T} > 1 - e^{-\gamma_1\beta_1(1,T)+\frac{1}{2}\gamma_1^2\Sigma_1(1)^2T}. \tag{47}$$

and for the second entity that

$$1 - e^{-\gamma_2\beta_2(\alpha^*,T)+\frac{1}{2}\gamma_2^2\Sigma_2(\alpha^*)^2T} > 1 - e^{-\gamma_2\beta_2(1,T)+\frac{1}{2}\gamma_2^2\Sigma_2(1)^2T}, \tag{48}$$

$$1 - e^{-\gamma_2\beta_2(\alpha^*,T)+\frac{1}{2}\gamma_2^2\Sigma_2(\alpha^*)^2T} > 1 - e^{-\gamma_2\beta_2(0,T)+\frac{1}{2}\gamma_2^2\Sigma_2(0)^2T}. \tag{49}$$

From the above equations for the first and second entities it is easy to verify that these conditions imply,

$$\beta_1(\alpha^*,T) - \beta_1(0,T) > \frac{1}{2}\gamma_1\left(\Sigma_1(\alpha^*)^2T - \Sigma_1(0)^2T\right), \tag{50}$$

$$\beta_1(\alpha^*,T) - \beta_1(1,T) > \frac{1}{2}\gamma_1\left(\Sigma_1(\alpha^*)^2T - \Sigma_1(1)^2T\right); \tag{51}$$

$$\beta_2(\alpha^*,T) - \beta_2(1,T) > \frac{1}{2}\gamma_2\left(\Sigma_2(\alpha^*)^2T - \Sigma_2(1)^2T\right), \tag{52}$$

$$\beta_2(\alpha^*,T) - \beta_2(0,T) > \frac{1}{2}\gamma_2\left(\Sigma_2(\alpha^*)^2T - \Sigma_2(0)^2T\right). \tag{53}$$

Equivalent, they can be written explicitly as

$$-\alpha^* c_v e^{rT} + \lambda(\alpha^*) y_v e^{u_v T} > \frac{1}{2}\gamma_1(2\lambda(\alpha^*)\rho_1\sigma_1\sigma_v + \lambda^2(\alpha^*)\sigma_v^2)T, \tag{54}$$

$$(1-\alpha^*) c_v e^{rT} - (1-\lambda(\alpha^*)) y_v e^{u_v T} > \frac{1}{2}\gamma_1(2(\lambda(\alpha^*)-1)\rho_1\sigma_1\sigma_v + (\lambda^2(\alpha^*)-1)\sigma_v^2)T; \tag{55}$$

$$-(1-\alpha^*) c_v e^{rT} + (1-\lambda(\alpha^*)) y_v e^{u_v T} > \frac{1}{2}\gamma_2(2(1-\lambda(\alpha^*))\rho_2\sigma_2\sigma_v + (1-\lambda(\alpha^*))^2\sigma_v^2)T, \tag{56}$$

$$\alpha^* c_v e^{rT} - \lambda(\alpha^*) y_v e^{u_v T} > \frac{1}{2}\gamma_2(-2\lambda(\alpha^*)\rho_2\sigma_2\sigma_v + \lambda^2(\alpha^*)\sigma_v^2)T. \tag{57}$$

From the above it is clear that for the first entity, if the extra enterprise risk generated by the project $(\Sigma_1(\alpha^*)^2 - \Sigma_1(0)^2)T = (2\lambda(\alpha^*)\rho_1\sigma_1\sigma_v + \lambda^2(\alpha^*)\sigma_v^2)T$ is positive, the risk adjusted contribution from participation in the project must be greater than $.5\gamma_1$:

$$\frac{\lambda(\alpha^*) y_v e^{u_v T} - \alpha^* c_v e^{rT}}{(2\lambda(\alpha^*)\rho_1\sigma_1\sigma_v + \lambda^2(\alpha^*)\sigma_v^2)T} > .5\gamma_1, \tag{58}$$

and similarly for the second entity that

$$\frac{(1-\lambda(\alpha^*)) y_v e^{u_v T} - (1-\alpha^*) c_v e^{rT}}{(2(1-\lambda(\alpha^*))\rho_2\sigma_2\sigma_v + (1-\lambda(\alpha^*))^2\sigma_v^2)T} > .5\gamma_2. \tag{59}$$

□

We note that $\lambda(\alpha^*)y_v e^{u_v T} - \alpha^* c_v e^{rT} = \beta_1(\alpha^*, T) - \beta_1(0, T)$ is the expected contribution of the shared participation in the joint venture and $(2\lambda(\alpha^*)\rho_1\sigma_1\sigma_v + \lambda^2(\alpha^*)\sigma_v^2)T = (\Sigma_1(\alpha^*)^2 - \Sigma_1(0)^2)T$ is the extra enterprise risk contributed by participation in the project (for entity one). Since γ_1 is the risk aversion coefficient, if the risk premium of the project is just sufficient to permit the party to enter it, then γ_1 is twice the project's risk premium per unit risk. Here its risk is measured by variance. Pratt's (1964) result in a single risky asset setting, unit time period, is derived by setting $\rho_1 = 0$,and $T = 1$. A similar argument and result can be made and derived for the second entity.

Finally we should note that the above conditions are sufficient for each party to gain from participation in the joint venture. Once they hold, the autarky solution is dominated by the position that requires taking a stake in the joint venture. It is not the case that overall risk is necessarily reduced but it is the case that expected utility is higher for each party, given the return and risk trade-offs. For the first entity, the expected variance of terminal wealth is

$$\Sigma_1(\alpha^*)^2 T = \left(\sigma_1^2 + 2\lambda(\alpha^*)\rho_1\sigma_1\sigma_v + \lambda^2(\alpha^*)\sigma_v^2\right) T. \tag{60}$$

For the second entity, the expected variance of terminal wealth is

$$\Sigma_2(\alpha^*)^2 = \left(\sigma_2^2 + 2(1-\lambda(\alpha^*))\rho_2\sigma_2\sigma_v + (1-\lambda(\alpha^*))^2\sigma_v^2\right)T. \qquad (61)$$

For both entities, these conditions can imply that expected variance of terminal wealth, can be clearly lower than that of the case where, the project is not pursued, i.e., $\Sigma_1(\alpha^*)^2 < \Sigma_1(0)^2$ and $\Sigma_2(\alpha^*)^2 < \Sigma_2(1)^2$, provided that $\rho_1 < -\lambda(\alpha^*)(\frac{\sigma_v}{\sigma_1})(.5)$ and $\rho_2 < -(1-\lambda(\alpha^*))(\frac{\sigma_v}{\sigma_2})(.5)$. So in this special case, the joint venture partners get the benefits of lower overall risk and higher expected utility, under the sharing agreement. However, there are limited opportunities for the entities to enjoy these dual benefits, higher expected utility and lower risk, as the condition, $0 < \lambda(\alpha^*) < 1$, must always hold for a viable joint venture.

In the case when ρ_1 is negative, the existing asset is negatively correlated with the new venture, investor one will be quite willing to participate in the joint venture as we can see if the extra enterprise risk $(2\lambda(\alpha^*)\rho_1\sigma_1\sigma_v + \lambda^2(\alpha^*)\sigma_v^2)T$ is negative, it is very easy for the expected contribution of the joint venture $\lambda(\alpha^*)y_v e^{u_v T} - \alpha^* c_v e^{rT}$ to be bigger than a negative number. Therefore, the interaction between the new venture and the existing programs and their combined effects on risk is key to the decision for participation in the joint venture.

Theorem 2.4. *For each venture partner, the decision not to take on the project solo, is driven by the fact that the excess cash flow caused by internalizing the project, is less than* $.5\gamma_i$ *the enterprise risk created.*

Proof. From equations (51), (53), (55) and (57), we have for the first entity

$$(1-\lambda(\alpha^*))y_v e^{u_v T}-(1-\alpha^*)c_v e^{rT} < -\frac{1}{2}\gamma_1(2(\lambda(\alpha^*)-1)\rho_1\sigma_1\sigma_v+(\lambda^2(\alpha^*)-1)\sigma_v^2)T, \tag{62}$$

and similarly for the second entity that

$$\lambda(\alpha^*)y_v e^{u_v T} - \alpha^* c_v e^{rT} < -\frac{1}{2}\gamma_2(-2\lambda(\alpha^*)\rho_2\sigma_2\sigma_v + \lambda^2(\alpha^*)\sigma_v^2)T. \tag{63}$$

For the first venture partner, the expected extra excess cashflow caused by project internalization is $\beta_1(1,T) - \beta_1(\alpha^*,T) = (1 - (\lambda(\alpha^*))y_v e^{u_v T} - (1 - \alpha^*)c_v e^{rT}$. The extra enterprise risk generated by accepting the entire project rather than a partial stake is $(\Sigma_1(1)^2 - \Sigma_1(\alpha^*)^2)T = .5\gamma_1\{-(1-\lambda(\alpha^*)^2)\sigma_v^2 + 2(1-\lambda(\alpha^*))\rho_1\sigma_1\sigma_v\}T$, and the implication for the Theorem is clear. □

3 Properties of the Optimal Risk Sharing Contract Design

One of the major features of joint ventures is that there is a tendency for them to be terminated by either or both parties mutually or perhaps under adversarial conditions by means of buyout or sale of an equity stake. Our model setup is devoid of any incentive considerations, except that of both parties wanting to maximize their individual welfare's. There is also no moral hazard and we have shown that the sharing solution dominates the autarky solution for each entity. There is no reason for our parties to deviate from the optimal rule. We first show particular hidden features of the sharing rule in equations (29) and (31).

We can rewrite equation (31) as

$$\lambda(\alpha^*) = \beta = \frac{\gamma_2\rho_2\sigma_2\sigma_v + \sigma_v^2\gamma_2 - \gamma_1\rho_1\sigma_1\sigma_v}{\sigma_v^2(\gamma_1 + \gamma_2)}. \tag{64}$$

The optimal dynamic sharing rule in our model differs from that of Bolton and Harris (2006) whose paper is somewhat close in spirit to ours. Neither partner's wealth or stake in the joint venture affects the global risk sharing rule, but their risk aversion preferences and the correlation of the cashflows of the new project with that of existing projects of the partners does affect the sharing rule. We can derive a myopic static sharing rule, that specifies the ratio of the agent's risk aversion parameter relative to the sum of the agent's and principal's risk aversion coefficients in our model by setting $\rho_2 = \rho_1 = 0$. That is $\lambda(\alpha^*) = \frac{\gamma_2}{\gamma_1+\gamma_2}$.

In this static, myopic setting, if the first entity is the party with the greater degree of risk aversion, it takes a lower share of risk in the new project. However, in the non-myopic case, this is no longer true. The more highly risk averse party may take the greater share of the new project's risk as its cashflows mesh nicely with cashflows from other existing projects. In fact, whether the more risk averse party takes on the larger or smaller share of the risks, depends critically on the equivalence between

$$\lambda(\alpha^*) > 1 - \lambda(\alpha^*)$$

and

$$\gamma_2\rho_2\sigma_2 - \gamma_1\rho_1\sigma_1 > \frac{(\gamma_1 - \gamma_2)\sigma_v}{2}.$$

Another significant point of difference is that in our model, even if the two parties have the same degree of risk aversion, there are still benefits to be shared, provided equation (29) and (31) hold.

The myopic rule of Bolton and Harris (2006) specifies[9]

$$\lambda_i = \frac{\gamma_i}{(\gamma_i + \gamma_j)}.$$

It is not surprising that in their setting, identical risk aversions between principal and agent lead to no sharing. They allow for the two parties to optimally invest in a common risky project and share the risk of the associated returns, but unlike our paper, outside projects are not considered. This

[9]Similar type myopic rules are mentioned by Savva and Scholtes (2006), Olsen and Osmundssen (2005) and others and are presented as the first best solutions when efforts are verifiable and when a stochastic cooperative game collapses to a deterministic cooperative game for instance. In all of these cases there is only one asset/good.

explains the fundamental difference. For instance, Olsen et al. (2005) generate a result similar in spirit to ours by considering effort sensitivity and its impact on incentives. Absent this condition, they also generate a Bolton and Harris (2006) result where there is no sharing.

In order to further facilitate the study of the stability property of optimal solutions, we provide the deltas in the following corollary.

Corollary 3.1. *Under the same conditions as in Theorem 2.2, we have the following derivatives:*

$$\frac{\partial\alpha^*}{\partial\gamma_1} = \frac{1}{\zeta(\gamma_1+\gamma_2)^2}\left(1+\frac{\rho_1\sigma_1}{\sigma_v}+\frac{\rho_2\sigma_2}{\sigma_v}\right), \quad \frac{\partial\alpha^*}{\partial\gamma_2} = \frac{1}{\zeta(\gamma_1+\gamma_2)^2}\left(1-\frac{\rho_1\sigma_1}{\sigma_v}+\frac{\rho_2\sigma_2}{\sigma_v}\right),$$
$$\frac{\partial\alpha^*}{\partial\sigma_1} = -\frac{\gamma_1\rho_1}{\zeta(\gamma_1+\gamma_2)\sigma_v}, \quad \frac{\partial\alpha^*}{\partial\sigma_2} = \frac{\gamma_2\rho_2}{\zeta(\gamma_1+\gamma_2)\sigma_v}, \quad \frac{\partial\alpha^*}{\partial\sigma_v} = \frac{-\gamma_2\rho_2\sigma_2+\gamma_1\rho_1\sigma_1}{\zeta(\gamma_1+\gamma_2)\sigma_v^2},$$
$$\frac{\partial\alpha^*}{\partial\rho_1} = -\frac{\gamma_1\sigma_1}{\zeta(\gamma_1+\gamma_2)\sigma_v}, \quad \frac{\partial\alpha^*}{\partial\rho_2} = \frac{\gamma_2\sigma_2}{\zeta(\gamma_1+\gamma_2)\sigma_v}, \quad \frac{\partial\alpha^*}{\partial c_v} = \frac{\partial\alpha^*}{\partial y_v} = \frac{\partial\alpha^*}{\partial\mu_v} = \frac{\partial\alpha^*}{\partial r} = 0,$$

and

$$\frac{\partial\lambda(\alpha^*)}{\partial\gamma_1} = \frac{1}{(\gamma_1+\gamma_2)^2}\left(1+\frac{\rho_1\sigma_1}{\sigma_v}+\frac{\rho_2\sigma_2}{\sigma_v}\right), \quad \frac{\partial\lambda(\alpha^*)}{\partial\gamma_2} = \frac{1}{(\gamma_1+\gamma_2)^2}\left(1-\frac{\rho_1\sigma_1}{\sigma_v}+\frac{\rho_2\sigma_2}{\sigma_v}\right),$$
$$\frac{\partial\lambda(\alpha^*)}{\partial\sigma_1} = -\frac{\gamma_1\rho_1}{(\gamma_1+\gamma_2)\sigma_v}, \quad \frac{\partial\lambda(\alpha^*)}{\partial\sigma_2} = \frac{\gamma_2\rho_2}{(\gamma_1+\gamma_2)\sigma_v}, \quad \frac{\partial\lambda(\alpha^*)}{\partial\sigma_v} = \frac{-\gamma_2\rho_2\sigma_2+\gamma_1\rho_1\sigma_1}{(\gamma_1+\gamma_2)\sigma_v^2},$$
$$\frac{\partial\lambda(\alpha^*)}{\partial\rho_1} = -\frac{\gamma_1\sigma_1}{(\gamma_1+\gamma_2)\sigma_v}, \quad \frac{\partial\lambda(\alpha^*)}{\partial\rho_2} = \frac{\gamma_2\sigma_2}{(\gamma_1+\gamma_2)\sigma_v},$$
$$\frac{\partial\lambda(\alpha^*)}{\partial c_v} = \frac{\partial\lambda(\alpha^*)}{\partial y_v} = \frac{\partial\lambda(\alpha^*)}{\partial\mu_v} = \frac{\partial\lambda(\alpha^*)}{\partial r} = 0.$$

Proof. Differentiating (31) to get the partial derivatives of $\lambda(\alpha^*)$ is an easy exercise. To derive the partial derivatives of α^*, we make use of implicit differentiation. Since $\alpha^* = \lambda^{-1}(\beta)$,

$$\frac{\partial \alpha^*}{\partial \gamma_1} = \frac{1}{\lambda'(\alpha^*)} \frac{\partial \lambda(\alpha^*)}{\partial \gamma_1} = \frac{1}{\zeta} \frac{\partial \lambda(\alpha^*)}{\partial \gamma_1}$$

$\diamond$

From the relations derived above we note that the optimal investment stake, α^*, and the optimal allocation share of the profits, $\lambda(\alpha^*)$, are invariant to the level of the interest rate, cost of the project, mean revenue and cost flows of the project. If the cash flows from the new project provide a natural hedge for existing cashflows for the first entity, the optimal share of its residual flows is decreased. Conversely if the cash flows from the new project do not provide a natural hedge for existing cashflows for the second entity, the optimal share of its residual flows to the first entity is increased. This is an eminently reasonable result. Assuming that $\zeta > 0$, the condition $\rho_1 < 0$ ensures that as the first entity receives the benefits of a natural cashflow hedge from its participation in the joint venture, that tends to reduce both its initial stake as well as cashflow allocation. This is only counterbalanced by the effect of the influence of ρ_2.

We will switch gear now to address another important issue: timing.

4 Optimal Timing of the JV investment

Suppose the opportunity of this joint venture exists for a certain period of time during which both investors can choose to enter the deal simultaneously. Then we have to look at their expected utilities as dynamic processes. If the prospects of both are going down, then the investors will not wait to enter the risk-sharing contract. This will be captured as 'super-martingale' conditions in the following analysis. First of all, in case both players enter the risk-sharing contract at time $t < T$, their wealth processes are

$$X_{1vt}(s) = \begin{cases} c_1 e^{rs} + y_1 e^{\mu_1 s} + \sigma_1 B_1(s), & s < t, \\ (c_1 e^{rt} - \alpha c_v) e^{r(s-t)} + y_1 e^{\mu_1 s} + \sigma_1 B_1(s) & \\ \qquad +\lambda(\alpha)(y_v e^{\mu_v (s-t)} + \sigma_v B_v(s)), & s \geq t; \end{cases}$$

$$X_{2vt}(s) = \begin{cases} c_2 e^{rs} + y_2 e^{\mu_2 s} + \sigma_2 B_2(s), & s < t, \\ (c_2 e^{rt} - (1-\alpha) c_v) e^{r(s-t)} + y_2 e^{\mu_2 s} + \sigma_2 B_2(s) & \\ \qquad +(1-\lambda(\alpha))(y_v e^{\mu_v (s-t)} + \sigma_v B_v(s)), & s \geq t. \end{cases}$$

In particular, at time $t = T$

$$
\begin{aligned}
X_{1vt}(T) &= (c_1 e^{rt} - \alpha c_v) e^{r(T-t)} + y_1 e^{\mu_1 T} + \sigma_1 B_1(T) \\
&\qquad + \lambda(\alpha)(y_v e^{\mu_v (T-t)} + \sigma_v B_v(T)), \\
&= \theta_1(\alpha, t, T) + \Sigma_1(\alpha) W_1(T), \\
X_{2vt}(T) &= (c_2 e^{rt} - (1-\alpha) c_v) e^{r(T-t)} + y_2 e^{\mu_2 T} + \sigma_2 B_2(T) \\
&\qquad + (1 - \lambda(\alpha))(y_v e^{\mu_v (T-t)} + \sigma_v B_v(T)), \\
&= \theta_2(\alpha, t, T) + \Sigma_2(\alpha) W_2(T),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1(\alpha, t, T) &= (c_1 e^{rt} - \alpha c_v) e^{r(T-t)} + y_1 e^{\mu_1 T} + \lambda(\alpha) y_v e^{\mu_v (T-t)}, \\
\theta_2(\alpha, t, T) &= (c_2 e^{rt} - (1-\alpha) c_v) e^{r(T-t)} + y_2 e^{\mu_2 T} + (1 - \lambda(\alpha)) y_v e^{\mu_v (T-t)},
\end{aligned}
$$

and $\Sigma_1(\alpha), \Sigma_2(\alpha), W_1(t), W_2(t)$ are defined in (7)-(10). Their expected utility functions at time t will be

$$
\begin{aligned}
u_1(\alpha, t) &= E\left[\, 1 - e^{-\gamma_1 X_{1vt}(T)} \,|\, \mathcal{F}_t \right] \\
&= 1 - e^{-\gamma_1 (\theta_1(\alpha, t, T) + \Sigma_1(\alpha) W_1(t)) + \frac{1}{2}\gamma_1^2 \Sigma_1(\alpha)^2 (T-t)}, \qquad (65) \\
u_2(\alpha, t) &= E\left[\, 1 - e^{-\gamma_2 X_{1vt}(T)} \,|\, \mathcal{F}_t \right] \\
&= 1 - e^{-\gamma_2 (\theta_2(\alpha, t, T) + \Sigma_2(\alpha) W_2(t)) + \frac{1}{2}\gamma_2^2 \Sigma_2(\alpha)^2 (T-t)}. \qquad (66)
\end{aligned}
$$

If $u_1(\alpha, t)$ and $u_2(\alpha, t)$ turn out to be super-martingales on $t \in [0, T]$ for all $\alpha \in (0, 1)$, then we call the risk-sharing contract *immediate*. In this case, the conditional mean is non-positive and both processes have a downward drift with respect to the contract inception time t, therefore it is better for both investors to enter the contract immediately.

Theorem 4.1. *Suppose all conditions in Theorem 2.1 are satisfied, and in addition for all* $\alpha \in (0, 1)$*:*

$$\max\left\{\frac{\alpha c_v r}{\lambda(\alpha) y_v \mu_v}, \frac{(1-\alpha) c_v r}{(1-\lambda(\alpha)) y_v \mu_v}\right\} \leq \min\left\{e^{(\mu_v - r)T}, 1\right\}. \tag{67}$$

Then $\lambda(\alpha)$ *is an immediate acceptable risk-sharing contract where* α^* *is the corresponding efficient risk-sharing plan.*

Proof. We first compute the derivatives

$$\begin{aligned}
\frac{\partial \theta_1(\alpha, t, T)}{\partial t} &= \alpha c_v r e^{r(T-t)} - \lambda(\alpha) y_v \mu_v e^{\mu_v (T-t)}, \\
\frac{\partial \theta_2(\alpha, t, T)}{\partial t} &= (1-\alpha) c_v r e^{r(T-t)} - (1-\lambda(\alpha)) y_v \mu_v e^{\mu_v (T-t)}.
\end{aligned}$$

Let

$$
\begin{aligned}
u_1(\alpha,t) &= 1 - e^{Y_1(t)},\\
&Y_1(t) = -\gamma_1(\theta_1(\alpha,t,T) + \Sigma_1(\alpha)W_1(t)) + \tfrac{1}{2}\gamma_1^2\Sigma_1(\alpha)^2(T-t);\\
u_2(\alpha,t) &= 1 - e^{Y_2(t)},\\
&Y_2(t) = -\gamma_2(\theta_2(\alpha,t,T) + \Sigma_2(\alpha)W_2(t)) + \tfrac{1}{2}\gamma_2^2\Sigma_2(\alpha)^2(T-t).
\end{aligned}
$$

Then

$$
\begin{aligned}
dY_1(t) &= -\gamma_1(\tfrac{\partial\theta_1(\alpha,t,T)}{\partial t}dt + \Sigma_1(\alpha)dW_1(t)) - \tfrac{1}{2}\gamma_1^2\Sigma_1(\alpha)^2dt,\\
dY_2(t) &= -\gamma_2(\tfrac{\partial\theta_2(\alpha,t,T)}{\partial t}dt + \Sigma_2(\alpha)dW_2(t)) - \tfrac{1}{2}\gamma_2^2\Sigma_2(\alpha)^2dt.
\end{aligned}
$$

By Itô's Lemma,

$$
\begin{aligned}
du_1(\alpha,t) &= -\left(e^{Y_1(t)}dY_1(t) + \tfrac{1}{2}e^{Y_1(t)}dY_1(t)dY_1(t)\right)\\
&= (1-u_1(\alpha,t))\gamma_1\left(\tfrac{\partial\theta_1(\alpha,t,T)}{\partial t}dt + \Sigma_1(\alpha)dW_1(t)\right).
\end{aligned}
$$

Similarly, we have

$$
du_2(\alpha,t) = (1-u_2(\alpha,t))\gamma_2\left(\tfrac{\partial\theta_2(\alpha,t,T)}{\partial t}dt + \Sigma_2(\alpha)dW_2(t)\right).
$$

Since $(1-u_1(\alpha,t))\gamma_1 > 0$ and $(1-u_2(\alpha,t))\gamma_2 > 0$, for $u_1(\alpha,t)$ and $u_2(\alpha,t)$

to be super-martingales, their $dt-$terms have to be non-positive

$$\tfrac{\partial\theta_1(\alpha,t,T)}{\partial t} \leq 0, \quad \tfrac{\partial\theta_2(\alpha,t,T)}{\partial t} \leq 0.$$

We have assumed that $c_v, r, y_v, \mu_v \geq 0$. Thus for all $\alpha \in (0,1)$

$$\begin{aligned}
\alpha c_v r e^{r(T-t)} - \lambda(\alpha) y_v \mu_v e^{\mu_v(T-t)} &\leq 0 \\
&\Leftrightarrow \frac{\alpha c_v r}{\lambda(\alpha) y_v \mu_v} \leq e^{(\mu_v - r)(T-t)}, \\
(1-\alpha) c_v r e^{r(T-t)} - (1-\lambda(\alpha)) y_v \mu_v e^{\mu_v(T-t)} &\leq 0 \\
&\Leftrightarrow \frac{(1-\alpha) c_v r}{(1-\lambda(\alpha)) y_v \mu_v} \leq e^{(\mu_v - r)(T-t)},
\end{aligned}$$

have to hold for all $t \in [0, T]$, and we obtain condition (67). $\diamond$

5 Conclusion

We have analyzed the problem of optimal risk-sharing in a dynamic continuous-time setting, where partners to a joint venture, only commit to the joint venture, if the venture is wealth improving and/or maximizes expected utility. The joint venture has risky cash flows and the primary impetus for risk sharing of the projects is that both parties can have greater expected utility. It is not because the partners cannot afford the project's costs separately. The joint project is such that expected utility decreases if the project is done as a solo venture by any partner. It also holds the possibility that for each partner expected variance of terminal wealth can also decrease, while expected utility will increase. The partners are risk averse but need not have the same degree of risk aversion. They however have individual projects outside of the common project and different levels of endowments. We have abstracted from bargaining and incentive issues to focus on the key elements of the optimal decision. In this regard we also do not consider institutional structures as both parties are assumed to act cooperatively and there is no hidden information and information symmetry on the joint project.

We are able to describe the optimal risk sharing contract and then derive the optimal shares of the cash flows that should be allocated to each partner. Our general sharing and optimal investment stake results generalize rules that are stated in the literature. Regardless of the functional form of the sharing rule, the optimal investment stake is a function of β (time-invariant) component and ζ (time-dependent component). We derive the comparative static properties of the optimal sharing rule and show that it is invariant to the initial endowments of the parties and the cash flows of the project. The optimal sharing rule only depends on the risk aversion parameters of the partners, volatilities of the cash flows of projects, their correlations and not the length of the investment horizon. Finally, we show that the conditions for beneficial risk-sharing is a form of Pratt's (1964) result, where multiple assets are explicitly considered. We also do some initial work on the optimal starting of the investment in the new project when both parties have expectations of declining future opportunities.

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