Infinite Horizon Optimal Search Problem with Hiring and Firing Options

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Abstract

As in the classic 'Secretary Problem': the candidates arrive sequentially. In this paper, they are represented with i.i.d. Itô diffusion processes. Two interwoven sequences of optimal stopping times are decided which signify the hiring and firing times of each candidate. The goal is to choose the stopping times to maximize expected sum of benefit and cost when the time horizon is infinite. The optimality conditions in terms of Verification Theorem, Least Superharmonic Majorant, and Variational Inequalities are given. The solution for the simple Brownian case with linear cost/benefit functions is calculated which results in a new two-one threshold strategy giving rise to the corresponding decisions of hiring/letting go-firing.

Key words: optimal stopping, search problem, secretary problem, marriage and divorce problem, entry and exit, sequential optimal stopping problem, invest and deinvest
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1 Introduction

1.1 Motivation for Applications

This paper studies an optimal stopping problem to sequentially hire and fire candidates who arrives at random times. The first candidate arrives at time 0. Two types of decisions can be made for this candidate over time depending on how the value process of this candidate evolves.

1. The candidate is hired at time τ_0 and then fired at time ζ_0 where $\tau_0 < \zeta_0$;
2. The candidate is let go without ever being hired. In this case, $\tau_0 = \zeta_0$ indicates the time when the decision is made not to make an offer.

After the final decision is made for the first candidate, all considerations about this candidate will cease, and the waiting for the second candidate starts. When the second candidate finally arrives after a waiting period s_1, the clock restarts where the same two types of decisions will have to be made for the second candidate with stopping times τ_1 and ζ_1. This process can repeat itself infinitely many times. If all the candidates have i.i.d. stochastic evolutions of their value processes, then the problem basically restarts at the arrival time of each candidate. The decision problem just being described can be viewed as a bridge between continuous-time sequential optimal stopping problems and the classic 'Secretary Problem'.

Now let us switch from the the view of hiring to that of a job seeker's perspective. Replace 'candidates' with 'job opportunities', then we can interpret τ_0 as the time the job seeker accepts the first job opportunity and ζ_0 as the time she quits it. For earlier work on structural job search models in discrete- and continuous-time respectively, see [7] and [5]. In general, this model can be applied to situations where two interwoven stopping time sequences have to be decided about a sequence of new 'opportunities'. In sociology, this could be applied to study of social relationships, for example, marriages and divorces. For financial applications, repeat decisions are often made for new investment opportunities, whether it be in the exploration of natural resources or in new ventures.

1.2 Mathematical Formulation of The Problem

Assume the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P^{s,\nu})$ owns the filtration $\mathcal{F}_t$ that satisfies the usual conditions. Let Y_t be an $\mathbb{R}$-valued Itô diffusion process

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \quad \forall t \geq s, \tag{1}$$

where B_t is an $\mathbb{R}$-valued Brownian motion and $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ satisfy

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq C|x - y|, \quad \forall x, y \in \mathbb{R},$$

for some constant $C \in \mathbb{R}$. The initial value Y_s of process Y_t is described by the distribution measure ν:

$$P^{s,\nu}(Y_s \in F) = \nu(F), \quad \forall F \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ denote the Borel sigma-algebra on $\mathbb{R}$. $P^{s,y}$ is the family of probability measures accompanying the strong Markov family Y_t with initial value y such that

$$P^{s,\nu}(F) = \int_{\mathbb{R}} P^{s,y}(F)\nu(dy), \quad \forall F \in \mathcal{F}.$$

It is understood that under $P^{s,y}$, Y_t follows the dynamics

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \quad \forall t \geq s; \quad Y_s = y.$$

Suppose the discount factor is a constant $r > 0$. The benefit rate function $f : \mathbb{R} \to \mathbb{R}$, the hiring cost function $N : \mathbb{R} \to \mathbb{R}$, and the firing cost function $K : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy that for any $s \geq 0$:

(a) $E^{s,y}[\int_s^\infty e^{-rt}|f(Y_t)|dt] < \infty$ for all $y \in \mathbb{R}$, where $E^{s,y}$ is the expectation under $P^{s,y}$, and

(b) the family $\{|N(Y_\alpha)| : \alpha \text{ is a stopping time}\}$ is uniformly integrable under $P^{s,y}$ for all $y \in \mathbb{R}$, and

(c) the family $\{|K(Y_\alpha)| : \alpha \text{ is a stopping time}\}$ is uniformly integrable under $P^{s,y}$ for all $y \in \mathbb{R}$.

Let (τ, ζ) be a pair of stopping times that decide the hiring and firing times of the candidate with value process Y_t, then we define the set of stopping times as

$$\mathcal{T}_s = \{(\tau, \zeta) : \tau \text{ and } \zeta \text{ are a.s. finite stopping times such that } s \leq \tau \leq \zeta\}. \tag{2}$$

The set $\{\tau = \zeta\}$ will be interpreted as those scenarios where decision is made to let go candidate Y_t without the possibility of her ever been hired, therefore the payoff function is zero on this set.[*] The optimal stopping problem for hiring and firing one candidate can be stated as

$$v(s, y) = \sup_{(\tau,\zeta)\in\mathcal{T}_s} E^{s,y}\left[\left(\int_\tau^\zeta e^{-rt}f(Y_t)dt + e^{-r\tau}N(Y_\tau) + e^{-r\zeta}K(Y_\zeta)\right)\mathbb{I}_{\{\tau<\zeta\}}\right]. \tag{3}$$

To simplify the notations, when $s = 0$, P^y stands for $P^{0,y}$, $\mathcal{T}$ stands for $\mathcal{T}_0$, and the above problem can be stated as

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^{y}\left[\left(\int_\tau^\zeta e^{-rt}f(Y_t)dt + e^{-r\tau}N(Y_\tau) + e^{-r\zeta}K(Y_\zeta)\right)\mathbb{I}_{\{\tau<\zeta\}}\right]. \tag{4}$$

To define the main problem with multiple i.i.d. candidates arriving sequentially, let Y_t^i, $i = 0, 1, 2, ...$, be i.i.d. processes with dynamics given by (1) and initial distribution ν, i.e.,

$$dY_t^i = \mu(Y_t^i)dt + \sigma(Y_t^i)dB_t^i, \quad \forall t \geq s, \quad i = 0, 1, 2, ...,$$

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[*] This definition prevents the interpretation of hiring and firing the candidate at the same time and making either an instantaneous profit or loss of $N(Y_\tau) + K(Y_\tau)$.

where B_t^i are independent $\mathbb{R}$-valued Brownian motions and

$$P^{s,\nu}(Y_s^i \in F) = \nu(F), \quad \forall F \in \mathcal{B}(\mathbb{R}), \quad i = 0, 1, 2, ...$$

Let $s_1, s_2, ...$ be a sequence of i.i.d. random variables whose moment generating function exist

$$\chi(u) = E[e^{-us_i}] < \infty, \quad i = 1, 2, ...$$

Suppose $s = 0$ and the initial candidate $(Y_t^0)_{t \geq 0}$ arrives at time zero. The hiring and firing times of this candidate are denoted as the pair of stopping times (τ_0, ζ_0). To make sure that the decision process is a restarting process at the arrival time of each new candidate and the new candidate will in no way interrupt the decision about the old candidate, assume that the second candidate arrives at a time with delay of s_1 after the first candidate has being fired: $T_1 = \zeta_0 + s_1$. The value process of the second candidate is thus

$$Z_t^1 = Y_t^1 \circ \theta_{T_1}^{-1}, \quad t \geq T_1,$$

where θ_s is the shift operator, i.e., $\theta_s : \Omega \to \Omega$ is $\mathcal{F}/\mathcal{F}$ measurable such that $Z_t(\omega) = Y_t \circ \theta_s(\omega), \forall \omega \in \Omega, \forall t \geq 0$. Note that

$$Z_t^1 = Y_{t-T_1}^1,$$

and effectively is the Y process shifted to the right to start at T_1, thus follows the dynamics

$$dZ_t^1 = \mu(Z_t^1)dt + \sigma(Z_t^1)dB_t^1, \quad \forall t \geq T_1.$$

Then the decision about the second candidate (τ_1, ζ_1) has to take place after the arrival time $\tau_1 \geq T_1$. Similarly, we denote the pairs of stopping times (τ_i, ζ_i) as hiring and firing times for i-th candidate with value processes Z_t^i, and the arrival times as $T_i = \zeta_{i-1} + s_i$, $i = 1, 2, ...$ Denote the set of stopping times under consideration for hiring and firing as

$$\mathcal{S} = \{\, (\tau_i, \zeta_i) : (\tau_i, \zeta_i) \text{ are a.s. finite stopping times such that } 0 \leq \tau_0 \leq \zeta_0 \leq T_1 \leq \tau_1 \leq \zeta_1 \leq T_2 \leq \tau_2 \leq ... \}.$$

Let $Z_t^0 = Y_t^0$. The objective of the main optimization problem is to maximize the total expected benefit

$$v(y) = \sup_{(\tau_i,\zeta_i)\in\mathcal{S}} E^y \left[\sum_{i=0}^{\infty} \left(\int_{\tau_i}^{\zeta_i} e^{-rt} f(Z_t^i)dt + e^{-r\tau_i} N(Z_{\tau_i}^i) + e^{-r\zeta_i} K(Z_{\zeta_i}^i) \right) \mathbb{I}_{\{\tau_i < \zeta_i\}} \right]. \tag{5}$$

The restarting nature of the problem can help to simplify the infinite sum above and reduce the problem to finding one pair of stopping times and our main problem (5) is equivalent to[†]

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^y \left[\left(\int_{\tau}^{\zeta} e^{-rt} f(Y_t)dt + e^{-r\tau} N(Y_\tau) + e^{-r\zeta} K(Y_\zeta) \right) \mathbb{I}_{\{\tau < \zeta\}} + e^{-r\zeta} \chi(r) \int_{\mathbb{R}} v(x)\nu(dx) \right], \tag{6}$$

where Y and $\mathcal{T} = \mathcal{T}_0$ are define in (1) and (2) as $s = 0$.

1.3 A Little History

The solution to the standard optimal stopping problem

$$v(y) = \sup_{\zeta\in\mathcal{R}_0} E^{0,y} \left[\int_0^{\zeta} e^{-rt} f(Y_t)dt + e^{-r\zeta} K(Y_\zeta) \right], \tag{7}$$

where $\mathcal{R}_s = \{\zeta : \zeta \text{ is a stopping time such that } \zeta \geq s\}$, is well-understood to be commonly a one-level threshold strategy. For reference books, see Karatzas and Shreve [6] for the probabilistic characterization

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[†] The details are shown in Appendix A.

of the value function as the Snell envelope, or see Øksendal [8] for the variational inequality with smooth pasting condition. Egami and Xu [4] studied the following problem (8) to optimally fire the current candidate with value Y_t at time ζ and hire the best candidate at time τ, where the new candidates are i.i.d random variables X^i arriving at Poisson jump times, and with a constant rate of search cost c.

$$v(y) = \sup_{(\zeta,\tau)\in\mathcal{T}} E^{0,y}\left[\int_0^{\zeta} e^{-rt}f(Y_t)dt + e^{-r\zeta}K(Y_\zeta) - \int_0^{\tau} ce^{-rt}dt + e^{-r\tau}X_\tau\right]. \tag{8}$$

The optimal solution turns out to be a one-one-level threshold strategy: one threshold for firing and another threshold for hiring. The Main Problem of this paper (6) is a further extension to (8) where the starting time is random and the new candidates are modeled by i.i.d. stochastic processes. The solution shown later in Section 3 turns out to be a two-one-level threshold strategy: one for hiring, one for letting go and another one for firing under the condition the candidate is hired. The Entry and Exit Problem studied in Brennan and Schwartz [2], Dixit [3], Brekke and Øksendal [1], decides the sequence of entry and exit times on a single opportunity and thus provides a one-one-level threshold strategy similar in spirit to the one candidate problem studied in Section 2. Shepp and Shiryaev [10] studied a very interesting problem with continuous hiring and firing problem in a different setting to this paper where multiple candidates are under simultaneous consideration.

2 Characterization of the Optimal Stopping Strategies for One Search Case

This section provides optimality conditions for a simpler problem where the search stops after just one candidate. The problem was stated in (4) and it is repeated below for convenience:

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^{y}\left[\left(\int_\tau^{\zeta} e^{-rt}f(Y_t)dt + e^{-r\tau}N(Y_\tau) + e^{-r\zeta}K(Y_\zeta)\right)\mathbb{I}_{\{\tau<\zeta\}}\right]. \tag{9}$$

This is simpler than our main problem (6) without the value function $v(\cdot)$ on the right-hand side, but still an extension of the classic problem (7) to require the decision for a pair of optimal stopping times. I take the standard approach of starting with a probabilistic characterization of martingale optimality conditions for (9), followed by PDE characterization under Markovian framework.

Theorem 2.1 (Verification Theorem) *Suppose there exists an adapted and continuous stochastic process* $(U_t^{s,y})_{t\geq s}$ *for which the following conditions hold for all* $y \in \mathbb{R}$*:*

a. $e^{-rt}U_t^{s,y} \geq \int_s^t e^{-ru}f(Y_u)du + e^{-rt}K(Y_t)$, $\forall t \geq s$, $P^{s,y}-a.s.$,

b. $e^{-rt}U_t^{s,y}$ *is a uniformly integrable supermartingale, and*

c. *there exists a stopping times* $\zeta^* \in \mathcal{R}_s$ *such that*

$$U_s^{s,y} = E^{s,y}\left[\int_s^{\zeta^*} e^{-rt}f(Y_t)dt + e^{-r\zeta^*}K(Y_{\zeta^*})\right];$$

and suppose there exists another adapted and continuous stochastic process $(V_t^{s,y})_{t\geq s}$ *for which the following conditions hold for all* $y \in \mathbb{R}$*:*

d. $e^{-rt}V_t^{0,y} \geq e^{-rt}N(Y_t) + e^{-rt}U_t^{t,Y_t}$, *for all* $t \geq 0$, $P^{0,y}-a.s.$,

e. $V_t^{0,y} \geq 0$, $\forall t \geq 0$, $P^{0,y}-a.s.$,

f. $e^{-rt}V_t^{0,y}$ *is a uniformly integrable supermartingale, and*

g. *there exists a stopping times* $\tau^* \in \mathcal{R}_0$ *such that*

$$V_0^{0,y} = E^{0,y}\left[\left(e^{-r\tau^*}N(Y_{\tau^*}) + e^{-r\tau^*}U_{\tau^*}^{\tau^*,Y_{\tau^*}}\right)\mathbb{I}_{\{\tau^*<\zeta^*\}}\right],$$

where $\zeta^* \in \mathcal{R}_{\tau^*}$ *is the optimal stopping time for achieving* $U^{\tau^*,Y_{\tau^*}}$ *in condition **(c)**. Then* $V_0^{0,y}$ *is the optimal value function in problem (9) and* (τ^*, ζ^*) *is a pair of optimal stopping times.*

Remark 2.2 *1. Process* $U_t^{s,y}$ *can be identified with the usual Snell envelope*

$$\operatorname*{ess\,sup}_{\zeta\in\mathcal{R}_s} E^{s,y}\left[\int_s^\zeta e^{-rt}f(Y_t)dt + e^{-r\zeta}K(Y_\zeta)\right]$$

and $V_t^{s,y}$ *with the new Snell envelope of (9).*

*2. The indicator function in the problem statement (9) introduces extra optionality (**e**) in the above Verification Theorem. This will appear again in the infinite search case in Theorem 3.1.*

Proof. In condition (**a**) and (**d**), the processes on both sides of the inequalities are continuous in time t, therefore if we replace it with random times $\tau \le \zeta$, the equalities still hold

$$e^{-r\zeta}U_\zeta^{\tau,Y_\tau} \ge \int_\tau^\zeta e^{-ru}f(Y_u)du + e^{-r\zeta}K(Y_\zeta), \quad P^{0,y} - a.s.,$$
$$e^{-r\tau}V_\tau^{0,y} \ge e^{-r\tau}N(Y_\tau) + e^{-r\tau}U_\tau^{\tau,Y_\tau}, \quad P^{0,y} - a.s.$$

Condition (**e**) implies that for any pair of stopping times $\tau \le \zeta$,

$$e^{-r\tau}V_\tau^{0,y} \ge 0, \quad P^{0,y} - a.s.$$

Apply Optional Sampling Theorem to conditions (**b**) and (**f**), and note that $\{\tau < \zeta\}$ is $\mathcal{F}_\tau$-measurable, we have

$$\begin{aligned}
V_0^{0,y} &\ge E^{0,y}\left[e^{-r\tau}V_\tau^{0,y}\right] = E^{0,y}\left[e^{-r\tau}V_\tau^{0,y}\mathbb{I}_{\{\tau<\zeta\}} + e^{-r\tau}V_\tau^{0,y}\mathbb{I}_{\{\tau=\zeta\}}\right] \\
&\ge E^{0,y}\left[\left(e^{-r\tau}N(Y_\tau) + e^{-r\tau}U_\tau^{\tau,Y_\tau}\right)\mathbb{I}_{\{\tau<\zeta\}}\right] \\
&\ge E^{0,y}\left[e^{-r\tau}N(Y_\tau)\mathbb{I}_{\{\tau<\zeta\}} + E^{0,y}\left[e^{-r\zeta}U_\zeta^{\tau,Y_\tau}\mathbb{I}_{\{\tau<\zeta\}} \,|\, \mathcal{F}_\tau\right]\right] \\
&\ge E^{0,y}\left[e^{-r\tau}N(Y_\tau)\mathbb{I}_{\{\tau<\zeta\}} + E^{0,y}\left[\left(\int_\tau^\zeta e^{-ru}f(Y_u)du + e^{-r\zeta}K(Y_\zeta)\right)\mathbb{I}_{\{\tau<\zeta\}} \,\Big|\, \mathcal{F}_\tau\right]\right] \\
&= E^{0,y}\left[\left(e^{-r\tau}N(Y_\tau) + \int_\tau^\zeta e^{-ru}f(Y_u)du + e^{-r\zeta}K(Y_\zeta)\right)\mathbb{I}_{\{\tau<\zeta\}}\right].
\end{aligned}$$

Taking supremum over all $(\tau, \zeta) \in \mathcal{T}$, we conclude one direction of the inequality $V_0^{0,y} \ge v(y)$. (**g**) implies the other direction of the inequality $V_0^{0,y} \le v(y)$. ◇

Corollary 2.3 (Least Superharmonic Majorant) *Suppose that* $u : \mathbb{R} \to \mathbb{R}$ *is a continuous function that satisfies the conditions*

h. $u(x) \ge K(x)$, $\forall\, x \in \mathbb{R}$,

i. *$e^{-rt}u(Y_t) + \int_0^t e^{-ru} f(Y_u)du$ is a $P^{0,y}$ uniformly integrable supermartingale for all $y \in \mathbb{R}$, and*

j. *for each $y \in \mathbb{R}$ there exists a stopping time $\zeta^* \in \mathcal{R}_0$ such that*

$$u(y) = E^{0,y}\left[\int_0^{\zeta^*} e^{-ru} f(Y_u)du + e^{-r\zeta^*}K(Y_{\zeta^*})\right].$$

If $u(x) + N(x) \leq 0$ for all $x \in \mathbb{R}$, then let $\tau^* = \zeta^* = 0$ and $v(y) = 0$ is the optimal value function in problem (9). Otherwise, suppose that $v : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the conditions

k. *$v(x) \geq (u(x) + N(x))^+$, $\forall\, x \in \mathcal{R}$,*

l. *$e^{-rt}v(Y_t)$ is a $P^{0,y}$ uniformly integrable supermartingale for all $y \in \mathbb{R}$, and*

m. *for each $y \in \mathbb{R}$ there exists a stopping time $\tau^* \in \mathcal{R}_0$ such that*

$$v(y) = E^{0,y}\left[\left(e^{-r\tau^*}N(Y_{\tau^*}) + e^{-r\tau^*}u(Y_{\tau^*})\right)\mathbb{I}_{\{\zeta^*>0\}}\right],$$

*where $\zeta^* \in \mathcal{R}_{\tau^*}$ is the solution to (**j**) for $u(Y_{\tau^*})$, i.e.,*

$$u(Y_{\tau^*}) = E^{0,Y_{\tau^*}}\left[\int_0^{\zeta^*} e^{-ru} f(Y_u)du + e^{-r\zeta^*}K(Y_{\zeta^*})\right].$$

Then $v(y)$ is the optimal value function in problem (9), and $(\tau^*, \zeta^* \circ \theta_{\tau^*})$ is a pair of optimal stopping times.

Remark 2.4 *Define $\mathcal{C}_u = \{x \in \mathbb{R} : u(x) > K(x)\}$, $\mathcal{F}_u = \{x \in \mathbb{R} : u(x) = K(x)\}$, $\mathcal{H}_v = \{x \in \mathbb{R} : v(x) = u(x) + N(x)\}$, $\mathcal{C}_v = \{x \in \mathbb{R} : v(x) > (u(x) + N(x))^+\}$ and $\mathcal{F}_v = \{x \in \mathbb{R} : v(x) = 0\}$.*

1. *When $u(x) + N(x) \leq 0$ for all $x \in \mathbb{R}$, it is never beneficial to continue because the payoff will always be negative, therefore $\mathcal{F}_v = \mathbb{R}$ and the candidate is let go without ever being hired $\tau^* = \zeta^* = 0$.*
2. *Otherwise, $\mathcal{F}_v = \emptyset$ because it is always possible to find a strategy to achieve positive payoff. In addition, $\{x \in \mathbb{R} : u(x) + N(x) < 0\} \subset \mathcal{C}_v$ for the same reason.*

Proof. Define $e^{-rt}U_t^{s,y} = \left(e^{-r(t-s)}u(Y_{t-s}) + \int_0^{t-s} e^{-ru} f(Y_u)du\right) \circ \theta_s$. It is obvious that conditions (**h**), (**i**), (**j**) imply conditions (**a**), (**b**), (**c**). Similarly, define $V_t^{0,y} = v(Y_t)$. Then conditions (**k**), (**l**) and (**m**) imply conditions (**d**), (**e**), (**f**) and (**g**). ◇

The next step is to provide an analytic characterization of optimality condition to (6) by variational inequalities coupled with 'smooth pasting' conditions. Its proof can be obtained from Corollary 2.3 and for consistency the format is a direct adaptation of Theorem 10.4.1 from Øksendal, B. (2003). Let

$$L = \mu(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} - r$$

be the partial differential operator characterizing the generator of discounted processes $e^{-rt}Y_t$. Denote $C^1(\mathbb{R})$ as the set of continuously differentiable functions and $C^2(\mathbb{R})$ as the set of twice continuously differentiable functions on $\mathbb{R}$.

Corollary 2.5 (Variational Inequality with Smooth Pasting)

- *Suppose $u : \mathbb{R} \to \mathbb{R}$ satisfies*
  1. *$u \in C^1(\mathbb{R})$,*

2. *$u(x) \geq K(x)$, $\forall\, x \in \mathbb{R}$,*
   Now define the continuation region as $\mathcal{C}_u = \{x \in \mathbb{R} : u(x) > K(x)\}$ and the firing region as $\mathcal{F}_u = \{x \in \mathbb{R} : u(x) = K(x)\}$,
3. *Y_t spends zero local time on $\partial\mathcal{C}_u$ a.s.: $E^{0,y}[\int_0^\infty \mathbb{I}_{\partial\mathcal{C}_u}(Y_t)dt] = 0$, $\forall y \in \mathbb{R}$,*
4. *$\partial\mathcal{C}_u$ is a Lipschitz surface,*
5. *$u \in C^2(\mathbb{R}\backslash\partial\mathcal{C}_u)$ and the second order derivative of u is locally bounded near $\partial\mathcal{C}_u$,*
6. *$Lu + f \leq 0$ on $\mathcal{F}_u$, and $Lu + f = 0$ on $\mathcal{C}_u$,*
   Now define $\zeta^* = \inf\{t \geq 0 : Y_t \notin \mathcal{C}_u\}$,
7. *$\zeta^* < \infty$, $P^{0,y}-$a.s.,*
8. *the family $\{u(Y_\zeta) : \zeta$ is a stopping time such that $\zeta \leq \zeta^*\}$ is uniformly integrable, $\forall y \in \mathbb{R}$.*

If $u(x) + N(x) \leq 0$ for all $x \in \mathbb{R}$, then let $\tau^* = \zeta^* = 0$ and $v(y) = 0$ is the optimal value function in problem (9). Otherwise,

- *suppose $v : \mathbb{R} \to \mathbb{R}$ satisfies*
  1. *$v \in C^1(\mathbb{R})$,*
  2. *$v(x) \geq (u(x) + N(x))^+$,$\forall\, x \in \mathbb{R}$,*
     Now define the continuation region as $\mathcal{C}_v = \{x \in \mathbb{R} : v(x) > (u(x) + N(x))^+\}$,
     and the hiring region as $\mathcal{H}_v = \{x \in \mathbb{R} : v(x) = u(x) + N(x)\}$,
  3. *Y_t spend zero local time on $\partial\mathcal{C}_v$ a.s.: $E^{0,y}[\int_0^\infty \mathbb{I}_{\partial\mathcal{C}_v}(Y_t)dt] = 0$, $\forall y \in \mathbb{R}$,*
  4. *$\partial\mathcal{C}_v$ is a Lipschitz surface,*
  5. *$v \in C^2(\mathbb{R}\backslash\partial\mathcal{C}_v)$ and the second order derivative of v is locally bounded near $\partial\mathcal{C}_v$,*
  6. *$Lv \leq 0$ on $\mathcal{H}_v$, and $Lv = 0$ on $\mathcal{C}_v$,*
     Now define $\tau^* = \inf\{t \geq 0 : Y_t \notin \mathcal{C}_v\}$,
  7. *$\tau^* < \infty$, $P^{0,y}-$a.s.,*
  8. *the family $\{v(Y_\tau) : \tau$ is a stopping time such that $\tau \leq \tau^*\}$ is uniformly integrable, $\forall y \in \mathbb{R}$;*

  $v(y)$ is the optimal value function in problem (9), and $(\tau^*, \zeta^* \circ \theta_{\tau^*})$ is a pair of optimal stopping times.

Example 2.6 (Simple Brownian Model with Linear Cost Functions) *Let $\mu(\cdot) = 0$, $\sigma(\cdot) = 1$, then $Y_t = B_t$ is a standard Brownian motion starting at y. The benefit and cost functions are assumed to be linear: $f(x) = ax, K(x) = bx, N(x) = cx$ where $a, b, c \in \mathbb{R}$ are constants. The discounting rate $r > 0$ is also a constant. Assume that the constants satisfy $a - br > 0, a + cr > 0, b + c > 0$[‡]. Then Problem (9) becomes*

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^{0,y}\left[\left(\int_\tau^\zeta e^{-rt}aB_t dt + e^{-r\tau}cB_\tau + e^{-r\zeta}bB_\zeta\right)\mathbb{I}_{\{\tau<\zeta\}}\right].$$

From Theorem 2.1 and Corollary 2.3, to obtain the solution we need to solve a pair of problems sequentially,

$$u(y) = \sup_\zeta E^{0,y}\left[\int_0^\zeta e^{-rt}aB_t dt + e^{-r\zeta}bB_\zeta\right],$$
$$v(y) = \sup_\tau E^{0,y}\left[\left(e^{-r\tau}cB_\tau + e^{-r\tau}u(B_\tau)\right)\mathbb{I}_{\{\zeta^*>0\}}\right].$$

The solution to the value function $u(y)$ is standard to compute when $a - br > 0$,

$$u(y) = \begin{cases} \frac{a}{r}y + \left(b - \frac{a}{r}\right)L^* e^{-(y-L^*)\sqrt{2r}}, & \text{for } y > L^*; \\ by, & \text{for } y \leq L^*, \end{cases}$$

[‡] The other cases for the relationship between constants can be similarly analyzed.

where $L^* = -\frac{1}{\sqrt{2r}}$. *The continuation region is* $\mathcal{C}_u = (L^*, \infty)$ *and the optimal stopping time is the first exit time of the continuation region* $\zeta^* = \inf\{t : B_t \le L^*\}$.

To find the solution for

$$v(y) = E^{0,y}\left[e^{-r\tau}\left(cB_\tau + u(B_\tau)\right)\mathbb{I}_{\{B_\tau > L^*\}}\right],$$

when $a + cr > 0$, *let us first guess the continuation region to be* $\mathcal{C}_v = (-\infty, U)$ *where* $U > 0$. *The first exit time is defined as* $\tau = \inf\{t : B_t \ge U\}$. *Then the value function related to the threshold strategy can be computed as*

$$\begin{aligned} J^U(y) &= \begin{cases} cy + u(y), & \text{for } y \ge U; \\ (cU + u(U))E^{0,y}[e^{-r\tau}], & \text{for } y < U. \end{cases} \\ &= \begin{cases} \left(c + \frac{a}{r}\right) y + \left(b - \frac{a}{r}\right) L^* e^{-(y-L^*)\sqrt{2r}}, & \text{for } y \ge U; \\ \left(\left(c + \frac{a}{r}\right) U + \left(b - \frac{a}{r}\right) L^* e^{-(U-L^*)\sqrt{2r}}\right) e^{-(U-y)\sqrt{2r}}, & \text{for } y < U. \end{cases} \end{aligned}$$

The smooth pasting condition $J^U \in C^1(\mathbb{R})$ *implies*

$$\begin{aligned} \left(c + \frac{a}{r}\right) + \left(b - \frac{a}{r}\right) e^{-(U-L^*)\sqrt{2r}} &= \sqrt{2r}\left[\left(c + \frac{a}{r}\right) U + \left(b - \frac{a}{r}\right) L^* e^{-(U-L^*)\sqrt{2r}}\right], \\ \Leftrightarrow 2\left(b - \frac{a}{r}\right) e^{-(U-L^*)\sqrt{2r}} &= \left(c + \frac{a}{r}\right)(\sqrt{2r}U - 1). \end{aligned}$$

When $b + c > 0$, *there is a unique solution* U^* *to the above equation which is greater than* L^*, *and the value function is*

$$v(y) = \begin{cases} \left(c + \frac{a}{r}\right) y + \left(b - \frac{a}{r}\right) L^* e^{-(y-L^*)\sqrt{2r}}, & \text{for } y \ge U^*; \\ \left(\left(c + \frac{a}{r}\right) U^* + \left(b - \frac{a}{r}\right) L^* e^{-(U^*-L^*)\sqrt{2r}}\right) e^{-(U^*-y)\sqrt{2r}}, & \text{for } y < U^*. \end{cases}$$

The optimal stopping time is $\tau^* = \inf\{t : B_t \ge U^*\}$. *The pair of optimal stopping time is thus* $(\tau^*, \zeta^* \circ \theta_{\tau^*})$, *where* $\zeta^* \circ \theta_{\tau^*} = \inf\{t \ge \tau^* : B_t \le L^*\}$.

3 Characterization of the Optimal Stopping Strategies for Infinite Search Case

This section solves the Main Problem (5) with infinite candidates. Its equivalent statement (6) is repeated here for convenience:

$$(10) \quad v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^y\left[\left(\int_\tau^\zeta e^{-rt} f(Y_t)dt + e^{-r\tau}N(Y_\tau) + e^{-r\zeta}K(Y_\zeta)\right)\mathbb{I}_{\{\tau<\zeta\}} + e^{-r\zeta}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right],$$

The derivation of the optimality theorems are similar to those in Section 2, therefore only their statements are provided.

Theorem 3.1 (Verification Theorem) *Suppose there exist adapted and continuous stochastic processes* $(U_t^{s,y})_{t\ge s}$ *and* $(V_t^{s,y})_{t\ge s}$ *for which the following conditions hold for all* $y \in \mathbb{R}$:

a. $e^{-rt}U_t^{s,y} \ge \int_s^t e^{-ru} f(Y_u)du + e^{-rt}K(Y_t) + e^{-rt}\chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)$, $\forall t \ge s$, $P^{s,y}-a.s.$,

b. $e^{-rt}U_t^{s,y}$ *is a uniformly integrable supermartingale, and*

c. *there exists a stopping times* $\zeta^* \in \mathcal{R}_s$ *such that*

$$U_s^{s,y} = E^{s,y}\left[\int_s^{\zeta^*} e^{-rt} f(Y_t)dt + e^{-r\zeta^*}K(Y_{\zeta^*}) + e^{-r\zeta^*}\chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)\right];$$

d. $e^{-rt}V_t^{0,y} \geq e^{-rt}N(Y_t) + e^{-rt}U_t^{t,Y_t}$, *$\forall t \geq 0$, $P^{0,y}-a.s.$*

e. $V_t^{0,y} \geq \chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)$, *$\forall t \geq 0$, $P^{0,y}-a.s.$*

f. $e^{-rt}V_t^{0,y}$ *is a uniformly integrable supermartingale, and*

g. *there exists a stopping times $\tau^* \in \mathcal{R}_0$ such that*

$$V_0^{0,y} = E^{0,y}\left[\left(e^{-r\tau^*}N(Y_{\tau^*}) + e^{-r\tau^*}U_{\tau^*}^{\tau^*,Y_{\tau^*}}\right)\mathbb{I}_{\{\tau^*<\zeta^*\}} + \left(e^{-r\tau^*}\chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)\right)\mathbb{I}_{\{\tau^*=\zeta^*\}}\right],$$

*where $\zeta^* \in \mathcal{R}_{\tau^*}$ is the optimal stopping time for achieving $U_{\tau^*}^{\tau^*,Y_{\tau^*}}$ in condition **(c)**. If $N(y) + U_0^{0,y} \leq \chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)$ for all $y \in \mathbb{R}$, then let $\tau^* = \zeta^* = 0$ and $v(y) = 0$ is the optimal value function in problem (10). Otherwise $V_0^{0,y}$ is the optimal value function in problem (10), and (τ^*, ζ^*) is a pair of optimal stopping times.*

Corollary 3.2 (Least Superharmonic Majorant) *Suppose that $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ are continuous functions that jointly satisfy the conditions*

h. $u(x) \geq K(x) + \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)$, $\forall\, x \in \mathbb{R}$,

i. $e^{-rt}u(Y_t) + \int_0^t e^{-ru}f(Y_u)du$ *is a $P^{0,y}$ uniformly integrable supermartingale for all $y \in \mathbb{R}$, and*

j. *for each $y \in \mathbb{R}$ there exists a stopping time $\zeta^* \in \mathcal{R}_0$ such that*

$$u(y) = E^{0,y}\left[\int_0^{\zeta^*} e^{-ru}f(Y_u)du + e^{-r\zeta^*}K(Y_{\zeta^*}) + e^{-r\zeta^*}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right];$$

k. $v(x) \geq \max\left\{u(x) + N(x),\, \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)\right\}$, $\forall\, x \in \mathbb{R}$,

l. $e^{-rt}v(Y_t)$ *is a $P^{0,y}$ uniformly integrable supermartingale for all $y \in \mathbb{R}$, and*

m. *for each $y \in \mathbb{R}$ there exists a stopping time $\tau^* \in \mathcal{R}_0$ such that*

$$v(y) = E^{0,y}\left[\left(e^{-r\tau^*}N(Y_{\tau^*}) + e^{-r\tau^*}u(Y_{\tau^*})\right)\mathbb{I}_{\{\zeta^*>0\}} + \left(e^{-r\tau^*}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right)\mathbb{I}_{\{\zeta^*=0\}}\right],$$

*where $\zeta^* \in \mathcal{T}_{\tau^*}$ is the solution to **(j)** for $u(Y_{\tau^*})$, i.e.,*

$$u(Y_{\tau^*}) = E^{0,Y_{\tau^*}}\left[\int_0^{\zeta^*} e^{-ru}f(Y_u)du + e^{-r\zeta^*}K(Y_{\zeta^*}) + e^{-r\zeta^*}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right].$$

If $u(x) + N(x) \leq \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)$ for all $x \in \mathbb{R}$, then let $\tau^* = \zeta^* = 0$ and $v(y) = 0$ is the optimal value function in problem (10). Otherwise, the optimal value function $v(y)$ is a positive function, and $(\tau^*, \zeta^* \circ \theta_{\tau^*})$ is a pair of optimal stopping times.

Corollary 3.3 (Variational Inequality with Smooth Pasting) *Suppose $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ satisfies*

1. *$u \in C^1(\mathbb{R})$ and $v \in C^1(\mathbb{R})$,*
2. *$u(x) \geq K(x) + \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)$, $\forall\, x \in \mathbb{R}$,*

   Now define the continuation region as $\mathcal{C}_u = \{x \in \mathbb{R} : u(x) > K(x) + \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)\}$ and the firing region as $\mathcal{F}_u = \{x \in \mathbb{R} : u(x) = K(x) + \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)\}$,

3. *$v(x) \geq \max\left\{u(x)+N(x),\, \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)\right\}$, $\forall x \in \mathbb{R}$,*

   Now define the continuation region as $\mathcal{C}_v = \left\{x \in \mathbb{R} : v(x) > \max\left\{u(x)+N(x),\, \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)\right\}\right\}$,
   the hiring region as $\mathcal{H}_v = \{x \in \mathbb{R} : v(x) = u(x)+N(x)\}$,
   and the firing region as $\mathcal{F}_v = \left\{x \in \mathbb{R} : v(x) = \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)\right\}$,

4. *Y_t spends zero local time on $\partial\mathcal{C}_u \cup \partial\mathcal{C}_v$ a.s.: $E^{0,y}[\int_0^\infty \mathbb{I}_{\partial\mathcal{C}_u}(Y_t)dt] = 0$ and $E^{0,y}[\int_0^\infty \mathbb{I}_{\partial\mathcal{C}_v}(Y_t)dt] = 0$, $\forall y \in \mathbb{R}$,*

5. *$\partial\mathcal{C}_u \cup \partial\mathcal{C}_v$ is a Lipschitz surface,*

6. *$u \in C^2(\mathbb{R}\backslash\partial\mathcal{C}_u)$ and the second order derivative of u is locally bounded near $\partial\mathcal{C}_u$, and $v \in C^2(\mathbb{R}\backslash\partial\mathcal{C}_v)$ and the second order derivative of v is locally bounded near $\partial\mathcal{C}_v$,*

7. *$Lu + f \leq 0$ on $\mathcal{F}_u$, and $Lu + f = 0$ on $\mathcal{C}_u$, $Lv \leq 0$ on $\mathcal{H}_v \cup \mathcal{F}_v$, and $Lv = 0$ on $\mathcal{C}_v$,*
   Now define $\zeta^* = \inf\{t \geq 0 : Y_t \notin \mathcal{C}_u\}$ and $\tau^* = \inf\{t \geq 0 : Y_t \notin \mathcal{C}_v\}$,

8. *$\zeta^* < \infty$ and $\tau^* < \infty$, $P^{0,y}-$a.s.,*

9. *the family $\{u(Y_\zeta) : \zeta$ is a stopping time such that $\zeta \leq \zeta^*\}$ is uniformly integrable, and the family $\{v(Y_\tau) : \tau$ is a stopping time such that $\tau \leq \tau^*\}$ is uniformly integrable, $\forall y \in \mathbb{R}$,*

If $u(x) + N(x) \leq \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)$ for all $x \in \mathbb{R}$, then let $\tau^* = \zeta^* = 0$ and $v(y) = 0$ is the optimal value function in problem (10). Otherwise, the optimal value function $v(y)$ is a positive function, and $(\tau^*, \zeta^* \circ \theta_{\tau^*})$ is a pair of optimal stopping times.

Example 3.4 (Simple Brownian Model with Linear Cost Functions Continued) *With the same assumptions as in Example 2.6: $\mu(\cdot) = 0$, $\sigma(\cdot) = 1$, $f(x) = ax, K(x) = bx, N(x) = cx$, where $a, b, c \in \mathbb{R}$ and $r > 0$ are constants. Assume that the constants satisfy $a - br > 0, a + cr > 0, b + c > 0$. The Main Problem (10) becomes*

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^y\left[\left(\int_\tau^\zeta e^{-rt}aB_t dt + e^{-r\tau}cB_\tau + e^{-r\zeta}bB_\zeta\right)\mathbb{I}_{\{\tau<\zeta\}} + e^{-r\zeta}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right].$$

Define the sequential optimization problems

$$u(y) = \sup_\zeta E^{0,y}\left[\int_0^\zeta e^{-rt}aB_t dt + e^{-r\zeta}bB_\zeta + e^{-r\zeta}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right], \tag{11}$$

$$v(y) = \sup_\tau E^{0,y}\left[\left(e^{-r\tau}cB_\tau + e^{-r\tau}u(B_\tau)\right)\mathbb{I}_{\{\zeta^*>0\}} + \left(e^{-r\tau}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right)\mathbb{I}_{\{\zeta^*=0\}}\right]. \tag{12}$$

Note that in Example 2.6, there is only one candidate, so giving up this candidate without hiring ($\zeta^* = 0$) means zero payoff. On the other side, waiting to hire the candidate at a reasonable level (U^*) will yield a positive payoff. The trade-off means that the candidate will not be let go without ever being hired, see Remark 2.4. In the current situation, letting go the current candidate and starting to wait for the next candidate restarts the problem with positive constant value $\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)$. Therefore, the candidate will be let go if the value is too low. It will turn out that two-one-level threshold strategies are optimal. For given constants $L, U, I \in \mathbb{R}$, define the hiring region as $\mathcal{H}_v = [U, \infty)$, continuation region as $\mathcal{C}_v = (L, U)$ and the firing region as $\mathcal{F}_v = (-\infty, L]$ associated to problem (12); continuation region as $\mathcal{C}_u = (I, \infty]$ and the firing region as $\mathcal{F}_u = (-\infty, I]$ associated to problem (11). The stopping times associated to the threshold strategies are $\zeta = \inf\{t : B_t \leq I\}$ and $\tau = \inf\{t : B_t \leq L$ or $B_t \geq U\}$, and the corresponding value functions are

$$u^I(y) = E^{0,y}\left[\int_0^\zeta e^{-rt}aB_t dt + e^{-r\zeta}bB_\zeta + e^{-r\zeta}\chi(r)\int_{\mathbb{R}} v^{L,U}(x)\nu(dx)\right], \tag{13}$$

$$v^{L,U}(y) = E^{0,y}\left[\left(e^{-r\tau}cB_\tau + e^{-r\tau}u^I(B_\tau)\right)\mathbb{I}_{\{B_\tau>I^*\}} + \left(e^{-r\tau}\chi(r)\int_{\mathbb{R}} v^{L,U}(x)\nu(dx)\right)\mathbb{I}_{\{B_\tau\leq I^*\}}\right]. \tag{14}$$

Define the constant restart value as

$$m^{L,U} = \chi(r)\int_{\mathbb{R}} v^{L,U}(x)\nu(dx). \tag{15}$$

Substituting $m^{L,U}$ *into (13) and (14), we have*

$$u^I(y) = E^{0,y}\left[\int_0^{\zeta} e^{-rt}aB_t dt + e^{-r\zeta}(bB_\zeta + m^{L,U})\right], \tag{16}$$

The solution to the value function $u^I(y)$ *is again standard to compute when* $\beta \overset{\Delta}{=} \frac{1}{\sqrt{2r}}\left(\frac{a}{r} - b\right) > 0$,

$$u^I(y) = \begin{cases} \frac{a}{r}y + \beta e^{-(y-I^*)\sqrt{2r}}, & \text{for } y > I^*; \\ by + m^{L,U}, & \text{for } y \le I^*, \end{cases}$$

where $I^* = \frac{m^{L,U}}{\beta\sqrt{2r}} - \frac{1}{\sqrt{2r}}$[§]*. When* $a + cr > 0$*, the solution to (14) is a two-threshold strategy where* $L^* \le I^* < U^*$*. For given pair* (L, U) *where* $L \le I^* < U$*, rewrite (14) as*

$$v^{L,U}(y) = \begin{cases} m^{L,U}, & \text{for } y \le L; \\ E^{0,y}\left[\left(e^{-r\tau}\left[cU + u^I(U)\right]\right)\mathbb{I}_{\{B_\tau = U\}} + e^{-r\tau}m^{L,U}\mathbb{I}_{\{B_\tau = L\}}\right], & \text{for } L < y < U; \\ cy + u^I(y), & \text{for } y \ge U. \end{cases}$$

In the case $L < y < U$,

$$\begin{aligned} v^{L,U}(y) &= \left[cU + u^I(U)\right]E^{0,y}\left[e^{-r\tau}\mathbb{I}_{\{B_\tau = U\}}\right] + m^{L,U}E^{0,y}\left[e^{-r\tau}\mathbb{I}_{\{B_\tau = L\}}\right], \\ &= \left[cU + u^I(U)\right]\frac{\sinh((y-L)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})} + m^{L,U}\frac{\sinh((U-y)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})}, \end{aligned}$$

where $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$*. Thus we have*

$$v^{L,U}(y) = \begin{cases} m^{L,U}, & \text{for } y \le L; \\ \left[cU + u^I(U)\right]\frac{\sinh((y-L)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})} + m^{L,U}\frac{\sinh((U-y)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})}, & \text{for } L < y < U; \\ \left(\frac{a}{r} + c\right)y + \beta e^{-(y-I^*)\sqrt{2r}}, & \text{for } y \ge U. \end{cases} \tag{17}$$

Continuous differentiability of $v^{L,U}(y)$ *at* $y = U$ *and* $y = L$ *yield*

$$\begin{aligned} \left(\frac{a}{r} + c\right) + \beta(-\sqrt{2r})e^{-(U-I^*)\sqrt{2r}} &= \left[cU + u^I(U)\right]\frac{\cosh((U-L)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})}\sqrt{2r} + m^{L,U}\frac{\cosh((U-U)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})}(-\sqrt{2r}), \\ 0 &= \left[cU + u^I(U)\right]\frac{\cosh((L-L)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})}\sqrt{2r} + m^{L,U}\frac{\cosh((U-L)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})}(-\sqrt{2r}). \end{aligned}$$

Define $\gamma \overset{\Delta}{=} \frac{1}{\sqrt{2r}}\left(\frac{a}{r} + c\right)$*, the above equations are equivalent to*

$$m^{L,U}\sinh((U-L)\sqrt{2r}) = \gamma - \beta e^{-(U-I^*)\sqrt{2r}}, \tag{18}$$

$$m^{L,U}\cosh((U-L)\sqrt{2r}) = \gamma\sqrt{2r}U + \beta e^{-(U-I^*)\sqrt{2r}}. \tag{19}$$

[§] Note that $(b - \frac{a}{r})I^* + m^{L,U} = \beta$.

From (15) and (17), we have

$$\begin{aligned} m^{L,U} &= \chi(r)\int_{\mathbb{R}} v^{L,U}(x)\nu(dx)\\ &= \chi(r)\Bigg\{\int_{(-\infty,L]} m^{L,U}\nu(dx) + \int_{[U,\infty)} \left(\left(\tfrac{a}{r}+c\right)x+\beta e^{-(x-I^*)\sqrt{2r}}\right)\nu(dx)\\ &\qquad + \int_{(L,U)}\left(\left[cU+u^I(U)\right]\tfrac{\sinh((x-L)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})} + m^{L,U}\tfrac{\sinh((U-x)\sqrt{2r})}{\sinh((U-L)\sqrt{2r})}\right)\nu(dx)\Bigg\}, \end{aligned}$$

where

$$\begin{aligned} &\Phi(L)=\int_{(-\infty,L]}\nu(dx), \qquad \Psi(U)=\int_{[U,\infty)}x\nu(dx), \qquad \Gamma(U)=\int_{[U,\infty)}e^{-x\sqrt{2r}}\nu(dx),\\ &M^-(L,U)=\frac{\int_{(L,U)}\sinh\big((U-x)\sqrt{2r}\big)\nu(dx)}{\sinh\big((U-L)\sqrt{2r}\big)}, \qquad M^+(L,U)=\frac{\int_{(L,U)}\sinh\big((x-L)\sqrt{2r}\big)\nu(dx)}{\sinh\big((U-L)\sqrt{2r}\big)}. \end{aligned}$$

It is equivalent to

$$m^{L,U}\left[\tfrac{1}{\chi(r)}-\Phi(L)-M^-(L,U)-\cosh((U-L)\sqrt{2r})M^+(L,U)\right]=\gamma\sqrt{2r}\Psi(U)+\beta e^{I^*\sqrt{2r}}\Gamma(U). \tag{20}$$

If there exist unique solutions (L^*,U^*,m^{L^*,U^*}) *to equations (18), (19) and (20) which satisfy* $L^*\le I^*=\frac{m^{L^*,U^*}}{\beta\sqrt{2r}}-\frac{1}{\sqrt{2r}}<U^*$*, then* L^*,U^*,I^* *provide the thresholds for the optimal strategy where (17) is the value function. Note that if the initial value of the future candidate follows a normal distribution with mean* $\bar{x}$ *and variance* $\bar{\sigma}$ *where* $\nu(dx)=\frac{1}{\sqrt{2\pi\bar{\sigma}^2}}e^{-\frac{(x-\bar{x})^2}{2\bar{\sigma}^2}}$*, then the functions* $\Phi(L),\Psi(U),\Gamma(U),M^-(L,U),M^+(L,U)$ *can be calculated very efficiently, and numerical solutions for (18), (19) and (20) are not hard to obtain.*

4 Future Work

There are many different directions to extend the current model and only a subset is listed below:

- New candidates arrive at random times $s_1, s_1+s_2, ...$ where s_i are i.i.d. random variables. The arrival of the new candidate can immediately affect the decision about the current candidate.
- Limit the model to finite number of candidates.
- Impose an exponentially distributed final time for all operations.
- A random number of candidates arrives simultaneously, but only one candidate will be under consideration.
- At any given time, up to a fixed number of candidates can be under consideration.

A Explanation of the Restarting Nature of Main Problem (6)

$$\begin{aligned} v(y)=\sup_{(\tau_i,\zeta_i)\in\mathcal{S}}\Bigg\{&E^y\left[\left(\int_{\tau_0}^{\zeta_0}e^{-rt}f(Z_t^0)dt+e^{-r\tau_0}N(Z_{\tau_0}^0)+e^{-r\zeta_0}K(Z_{\zeta_0}^0)\right)\mathbb{I}_{\{\tau_0<\zeta_0\}}\right]\\ &+E^y\left[E^{Z_{T_1}^1}\left[\sum_{i=1}^{\infty}\left(\int_{\tau_i}^{\zeta_i}e^{-rt}f(Z_t^i)dt+e^{-r\tau_i}N(Z_{\tau_i}^i)+e^{-r\zeta_i}K(Z_{\zeta_i}^i)\right)\mathbb{I}_{\{\tau_i<\zeta_i\}}\Big|\mathcal{F}_{T_1}\right]\right]\Bigg\} \end{aligned}$$

$$\begin{aligned}
&= \sup_{(\tau_0,\zeta_0)\in\mathcal{S}} \Bigg\{ E^y \Bigg[\bigg(\int_{\tau_0}^{\zeta_0} e^{-rt} f(Y_t^0)dt + e^{-r\tau_0} N(Y_{\tau_0}^0) + e^{-r\zeta_0} K(Y_{\zeta_0}^0) \bigg) \mathbb{I}_{\{\tau_0<\zeta_0\}} \Bigg] \\
&\qquad + E^y \Bigg[\text{ess} \sup_{(\tau_i,\zeta_i)\in\mathcal{S}} E^{Y_{T_1}^1\circ\theta_{T_1}^{-1}} \Bigg[\sum_{i=1}^{\infty} \bigg(\int_{\tau_i}^{\zeta_i} e^{-rt} f(Y_t^i\circ\theta_{T_1}^{-1})dt + e^{-r\tau_i} N(Y_{\tau_i}^i\circ\theta_{T_1}^{-1}) \\
&\qquad + e^{-r\zeta_i} K(Y_{\zeta_i}^i\circ\theta_{T_1}^{-1})\big) \, \mathbb{I}_{\{\tau_i\circ\theta_{T_1}<\zeta_i\circ\theta_{T_1}\}} \Bigg]\Bigg]\Bigg\} \\
&= \sup_{(\tau_0,\zeta_0)\in\mathcal{S}} \Bigg\{ E^y \Bigg[\bigg(\int_{\tau_0}^{\zeta_0} e^{-rt} f(Y_t^0)dt + e^{-r\tau_0} N(Y_{\tau_0}^0) + e^{-r\zeta_0} K(Y_{\zeta_0}^0) \bigg) \mathbb{I}_{\{\tau_0<\zeta_0\}} \Bigg] \\
&\qquad + E^y \Bigg[\text{ess} \sup_{(\tau_i\circ\theta_{T_1},\zeta_i\circ\theta_{T_1})} E^{Y_0^1} \Bigg[\sum_{i=1}^{\infty} \bigg(\int_{\tau_i\circ\theta_{T_1}}^{\zeta_i\circ\theta_{T_1}} e^{-r(T_1+t)} f(Y_t^i)dt + e^{-r(T_1+\tau_i\circ\theta_{T_1})} N(Y_{\tau_i}^i) \\
&\qquad + e^{-r(T_1+\zeta_i\circ\theta_{T_1})} K(Y_{\zeta_i}^i) \bigg) \, \mathbb{I}_{\{\tau_i\circ\theta_{T_1}<\zeta_i\circ\theta_{T_1}\}} \Bigg]\Bigg]\Bigg\} \\
&= \sup_{(\tau_0,\zeta_0)\in\mathcal{S}} \Bigg\{ E^y \Bigg[\bigg(\int_{\tau_0}^{\zeta_0} e^{-rt} f(Y_t^0)dt + e^{-r\tau_0} N(Y_{\tau_0}^0) + e^{-r\zeta_0} K(Y_{\zeta_0}^0) \bigg) \mathbb{I}_{\{\tau_0<\zeta_0\}} \Bigg] \\
&+ E^y \Bigg[e^{-rT_1} \text{ess} \sup_{(\tau_i,\zeta_i)\in\mathcal{S}} E^{Y_0^0} \Bigg[\sum_{i=0}^{\infty} \bigg(\int_{\tau_i}^{\zeta_i} e^{-rt} f(Y_t^i)dt + e^{-rt} N(Y_{\tau_i}^i) + e^{-rt} K(Y_{\zeta_i}^i) \bigg) \mathbb{I}_{\{\tau_i<\zeta_i\}} \Bigg]\Bigg]\Bigg\} \\
&= \sup_{(\tau_0,\zeta_0)\in\mathcal{S}} \Bigg\{ E^y \Bigg[\bigg(\int_{\tau_0}^{\zeta_0} e^{-rt} f(Y_t^0)dt + e^{-r\tau_0} N(Y_{\tau_0}^0) + e^{-r\zeta_0} K(Y_{\zeta_0}^0) \bigg) \mathbb{I}_{\{\tau_0<\zeta_0\}} \Bigg] + E^y \left[e^{-rT_1} v(Y_0^0) \right] \Bigg\} \\
&= \sup_{(\tau_0,\zeta_0)\in\mathcal{S}} E^y \Bigg[\bigg(\int_{\tau_0}^{\zeta_0} e^{-rt} f(Y_t^0)dt + e^{-r\tau_0} N(Y_{\tau_0}^0) + e^{-r\zeta_0} K(Y_{\zeta_0}^0) \bigg) \mathbb{I}_{\{\tau_0<\zeta_0\}} + e^{-r\zeta_0} \chi(r) \int_{\mathbb{R}} v(x)\nu(dx) \Bigg].
\end{aligned}$$

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