Understanding Fundamental Ideas in Mathematics at a Deep Level

A number of mathematicians and mathematics education researchers have recognized the special nature of the mathematical knowledge needed for K-12 teaching and its implications for the mathematical preparation of teachers. In particular, the interviews with Chinese elementary teachers in Liping Ma's 1999 book Knowing and Teaching Elementary Mathematics awakened many mathematicians to this issue and its mathematical substance. The mathematics to which U.S. schoolchildren are exposed from preschool through eighth grade has many aspects. However, at the heart of preschool, elementary school, and middle school mathematics is the set of concepts associated with the term *number*. Children learn to count, and they learn to keep track of their counting by writing numerals for the natural numbers. They learn to add, subtract, multiply, and divide whole numbers, and later in elementary school they learn to perform these same operations with common fractions and decimal fractions. They use numbers in measuring a variety of quantities, including the lengths, areas, and volumes of geometric figures. From various sources, children collect data that they learn to represent and analyze using numerical methods. The study of algebra begins as they observe how numbers form systems and as they generalize number patterns. Mathematics is often taught in elementary school as a set of algorithms without developing the conceptual understanding needed to move to higher levels. US teachers often have very good procedural understanding of the arithmetic of integers, fractions and decimals, yet a profound conceptual understanding in teachers is essential, as they must provide their students with this needed understanding for reaching algebra and even higher levels of mathematical thinking.

This seminar aims to show participants that deep understanding of elementary ideas like place value is attainable in elementary classrooms, and that one way to cultivate this understanding is through irresistible problems.

The seminar takes the position that learning mathematics can be motivated by interesting problems. The trick is to come up with problems whose solutions either require or strongly motivate the development of the area of mathematics to be learned. One could also take the position that mathematics is **about** problem solving.

Fortunately, there are plenty of arithmetic and geometric problems that motivate the need for algebraic thinking. And on top of that, solving interesting mathematical problems in an appropriate social setting can really be fun. Have a look at the problems below. You might not be able to solve any of them on the fly. But with two or three partner teachers, you can solve them all. Some of the problems below can be used to build entire lessons. For example, the first problem could motivate the entire section on place-value.

List of Topics

1. Place Value
2. Representing Rational Numbers, syntax vs. semantics, $.\overline{9} = 1$.
3. Unit Cube Problems and their generalizations
4. Euclidean Algorithm, decanting, lcd, gcd
5. Number Representations and arithmetic; Fibonacci, Cantor, etc.
6. Bug in the Plane Problems
7. Perfect Card Trick
8. Euler's Formula, using Zome Tools
9. Single Pile Nim Games, and Bouton's Nim, Puppies and Kittens
10. Solving Linear and Quadratic Equations in Z_7

Problems To Try

1. **X'ing digits.** Consider the number

   $$N = 123456789101112\ldots5960.$$

   What is the largest number that can be produced by crossing out exactly 100 digits of N?

2. Two positive integers a and b satisfy

   $$a + \frac{b}{b + \frac{1}{a+2}} = \frac{38}{13}.$$

   What is $a + b$?

3. Notice that when the digits of 3981 are reversed, we get 1893. The difference is $3981 - 1893 = 2088$. How many of the numbers in the range 1000 to 9999 are exactly 2088 larger than their four-digit reversal?

4. Find six different decimal digits a, b, c, d, e, f so that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1$, but the sum is as large as possible.

5. A rectangle with integral area has a perimeter of 13. How many different shapes are possible? Prove your answer.

6. The number $N = 123456789101112\ldots999$ is formed by stringing together all the numbers from 1 to 999. What is the product of the 2007^{th} and 2008^{th} digits of N?

7. Using the ten digits each exactly once, create 3 numbers A, B, and C, such that $A + B = C$.

8. Using all nonzero digits each once, build two numbers A and B so that $A \cdot B$ is as large as possible.

9. From a 19th century trade card advertising *Bassetts Horehound Troches*, a remedy for coughs and colds: A man buys 20 pencils for 20 cents and gets three kinds of pencils in return. Some of the pencils cost 4 cents each, some are two for a penny and the rest are four for a penny. How many pencils of each type does the man get?

References

[1] Taking Place Value Seriously: Arithmetic, Estimation and Algebra (PDF 652KB)
http://www.maa.org/PMET/resources/PVHoweEpp-Nov2008.pdf An essay by Roger Howe, Yale University and Susanna Epp, DePaul University. Arithmetic, first of whole numbers, then of decimal and common fractions, and later of rational expressions and functions, is a central theme in school mathematics. This essay attempts to point out ways to make the study of arithmetic more unified and more conceptual through systematic emphasis of place value structure in the decimal number system.

[2] Ma, Liping (1999). Knowing and teaching elementary mathematics: Teachers' understanding of mathematics in China and the United States. Mahwah, NJ: Lawrence Erlbaum Associates. Taken from an online review: 'Elementary school teachers are expected to teach almost everything: math, reading, science, social studies, and writing; along with nurturing, soothing, and encouraging. It's not an easy job. It's also hard to be an expert in any one piece of the job. But now, many are hearing that we're losing the "math race" to other countries. The drums of "teacher competency" are booming... and any wise teacher knows where the drum sticks will be landing next!"

[3] The Major Topics of School Algebra by Wilfried Schmid and H. Wu
http://math.berkeley.edu/~wu/NMPalgebra7.pdf An essay listing the topics in high school algebra essential for advanced mathematics in college.

[4] Arithmetic for Parents: A Book for Grownups about Children's Mathematics by Ron Aharoni. Online review: 'Ron Aharoni writes clearly and deeply about the crucial concepts of fundamental maths, how to teach them and how not to teach them. He explains the layered and subtle structure of elementary maths and how missing a layer can lead to frustration and maths anxiety. "There's no royal road the maths", an Euclidian quote he emphasizes which summarizes well the message in this book. I'm not sure the book is for "Parents" as its title suggests, but I highly recommend it for both lovers and "haters" of maths, regardless of their "parental status." Looking forward to Ron's next book. '

[5] Here's a collection of units written by New Haven mathematics teachers as part of the Yale-New Haven Teachers Institute:
http://www.yale.edu/ynhti/curriculum/units/2004/5/