Problems with Cubes

Let n be a positive integer. Consider the $n \times n \times n$ cube. Paint the exterior of the cube. Then cut the cube into n^3 unit cubes. Let $N_0, N_1, N_2, \ldots, N_6$ denote the number of cubes with $0, 1, 2, \ldots, 6$ painted faces. Of course $N_6 = 0$ unless $n = 1$, and then $N_6 = 1$. It is not hard to see that, for each n, $\sum_{i=0}^{6} N_i = n^3$. For example, when $n = 3$, we have $N_0 = 1, N_1 = 6, , N_2 = 12, N_3 = 8$, and $N_4 = N_5 = N_6 = 0$.

1. Find the number of cubes with some painted faces as a function of n. Note that for $n = 3$, the number is $27 - 1 = 26$.

2. Suppose that every pair of interior faces are glued together so that each pair of faces requires one unit of glue. How many square units of glue is needed? Examine this for $n = 4, 5$, and 6.

3. What is the minimum amount of glue needed to hold together all n^3 cubes, for $n = 2, 3, 4, 5$. This brings up the issue of whether a cube is rigid if the surface is rigid. We can try this problem under both assumptions, one where we simply need make the surface rigid and the other where every cube must get some glue. Try these for $n = 3$ where only the surface of the cube need be rigid.

4. Suppose two non-adjacent faces of the big cube are painted red and the other four faces painted black. Let R denote the number of unit cubes with some red faces and B the number of unit cubes with some black faces. Find an n for which $B - R = 390$.

5. (2008 Mathcounts) A $12 \times 12 \times 12$ cube is built using a $10 \times 10 \times 10$ cube and a bunch of $2 \times 2 \times 2$ cubes. How many $2 \times 2 \times 2$ cubes are needed? A $(n + 2) \times (n + 2) \times (n + 2)$ cube is built using a $n \times n \times n$ cube, a bunch of $2 \times 2 \times 2$ cubes and a few unit cubes. How many unit cubes and $2 \times 2 \times 2$ cubes are needed? Your answer may depend on the oddness or evenness of n.

6. You're given 8 unpainted cubes. Can you paint the faces with two colors, red and blue, so that when you're done, you can assemble both an all red cube and an all blue cube?

7. You're given 27 unpainted cubes. Can you paint the faces with three colors, red, white, and blue, so that when you're done, you can assemble an all red $3 \times 3 \times 3$ cube, an all white $3 \times 3 \times 3$ cube and an all blue $3 \times 3 \times 3$ cube?

8. You're given 64 unpainted cubes. Can you paint the faces with four colors, red, white, green and blue, so that when you're done, you can assemble cubes of all four colors?

9. Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. The number of unit cubes with some red paint is found to be exactly 200 larger than the number of cubes with some black paint. How many cubes have no paint at all?

10. Suppose some faces of an $n \times n \times n$ wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Let R denote the number of cubes with some red paint and B the number of cubes with some black paint. What is the least value of n for which $B+R$ is a multiple of 100? Find the next five values of n for which $B+R$ is a multiple of 100. In each case decide how the faces of the big cube are painted.