1. **The Four 4's problem.** We have four copies of the digit 4 to use in this problem. The idea is to combine them in different ways to count to 100. We'll try to construct each number, 1, 2, 3, etc. up to 100 using four 4's, and when we can't construct a number, we'll allow ourselves to use five 4's. The operations we can use are the usual arithmetic operations, plus, minus, times, and divides $+, -, \times, \div$. We also allow ourselves *concatenation* . For example, we can build the number $4 * 4 = 44$ from two 4's. Also, note that $(4 \times 4) * 4 = 164$. When there is not possible confusion, we write just 44 instead of $4 * 4$. Here are a few examples to get you started. $1 = 44 \div 44$, $2 = (4 \div 4) + (4 \div 4)$, and $3 = (4 + 4 + 4) \div 4$. Be sure you use parentheses to make your expressions clearly defined. The important thing here is to see which numbers cannot be constructed with four 4's.

2. **X'ing digits.** Consider the number

   $$N = 123456789101112131415161718192021222324252627282930\ldots5960$$

   obtained by writing the numbers from 1 to 60 next to one another. What is the largest number that can be produced by crossing out 100 digits? You are not allowed to rearrange the digits that you don't cross out.

3. Juan and Thu are both smart chocolate-lovers. There are four bars of chocolate of sizes 250 grams, 300 grams, 400 grams and 600 grams. Adam chooses first and starts eating at a uniform (=constant) rate. As soon as Juan chooses, Thu gets to chose which chocolate bar to start on, and she eats at the same uniform rate as Adam. As soon as one of them finishes, that person chooses again and again eats at the same rate. Who gets the most chocolate. Explain how they can achieve it.

4. A rectangular block of size $3 \times 4 \times 5$ is built from 60 unit cubes. How many of the 60 cubes can be seen from the outside?

5. Find four different digits selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ to build two fractions each with a single digit numerator and single digit denominator so that the sum of the two fractions is less than 1 but as large as possible otherwise.

6. The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Use this information to find the exact location of all the mines.

	1		1		2		2		1
1		2		3		2		3	
	3		2		3		2		2
1		3		2		1		3	
	3		2		1		1		2
2		4		1		1		2	
	3		3		2		3		1
1		2		2		3		2	

7. A *special-8* number is one whose decimal representation consists entirely of 0's and 8's. For example 0.8808 and $0.\overline{08}$ are special numbers. What is the fewest special numbers whose sum is 1.

8. The numbers $1, 2, 3, 7, 8, 9$ are arranged in a multiplication table, with three along the top and the other three down the column. The multiplication table is completed and the sum of the nine entries is tabulated. What is the largest possible sum obtainable. Hint: can you find a way to think of this sum as the area of a rectangle?

$\times$	a	b	c
d			
e			
f			

9. A *special-3* number is one whose decimal representation consists entirely of 0's and 3's. For example 0.3033 and $0.\overline{03}$ are special numbers. What is the fewest special numbers whose sum is 1.

10. Answer the same question about *special-6*'s and *special-7*'s.

11. Definitions: All variables stand for positive integers unless otherwise specified. We consider symmetric solutions, like $1^2 + 2^2 + 3^2 = 14$ and $2^2 + 3^2 + 1^2 = 14$, to be two examples of the same solution. So we'd say there's just one positive integer solution to the equation $a^2+b^2+c^2 = 14$, and we'd write it in increasing order as $(1, 2, 3)$. And a warning: when the question asks to list all solutions, it may well be that there are none!

    (a) List all (positive integer, $a \leq b$) solutions to $a^2 + b^2 = 5^2$.

    (b) List all solutions to $a^2 + b^2 = 11$.

(c) When $a^2 + b^2$ is divided by 4, what are all possible remainders?

(d) List all solutions to $a^2 + b^2 = 1003$.

(e) What are the possible remainders when a^2 is divided by 7?

(f) List all solutions to $a^2 + b^2 = 65$.

(g) List all solutions to $a^2 + b^2 = 65^2$.

(h) Give an example of positive integers a and b such that $a^2 + b^2$ is divisible by 7.

(i) If $a^2 + b^2$ is divisible by 7, what is the largest number that $a^2 + b^2$ must also be divisible by?

(j) List all solutions to $a^2 + b^2 = 1001$.

(k) How many solutions are there to $a^2 + b^2 = 221^2$?

(l) What is the smallest positive odd integer n such that $n^2 + b^2 = c^2$ has exactly two positive integer solutions with $n < b$?

12. The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Use this information to find the exact location of all the mines. A square can have more than one mine.

	1		1		2		2		1
1		2		3		2		3	
	4		2		4		2		2
2		4		3		2		3	
	4		2		2		1		2
2		4		1		2		2	
	3		3		3		4		1
1		2		2		4		2	

13. Let N be the huge number

$$N = 123456789101112\ldots 999$$

obtained by writing down, in order, the representation of the first 999 positive integers.

(a) How many digits does N have?

(b) How many times does the digit 6 appear in N?

(c) What is the product of the 2009^{th} digit and the 2010^{th} digit of N?

14. **Inserting Plus Signs.**
Plus signs can be inserted in

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

in any of six positions. For example, we could put '+' signs in the second, fourth, and sixth places to get

$$12 + 34 + 56 + 7 = 109.$$

(a) Can one or more plus signs be inserted to achieve the number 100? If so, in how many ways can this be done?

(b) Suppose either a $+$ or a $-$ sign MUST be inserted in each position. What are the numbers that could result?

(c) Suppose a $+$ sign or a $-$ sign may be inserted in each of the eight positions of $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$. Can the number 100 be achieved?

(d) Now back to the first problem. How many numbers can be achieved by putting $+$ signs into $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$?