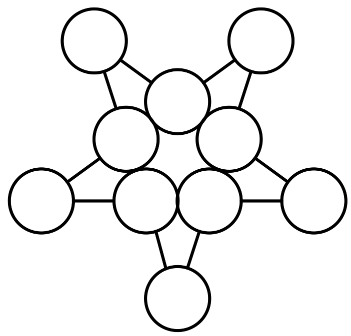
Magic Figures and Difference Triangles

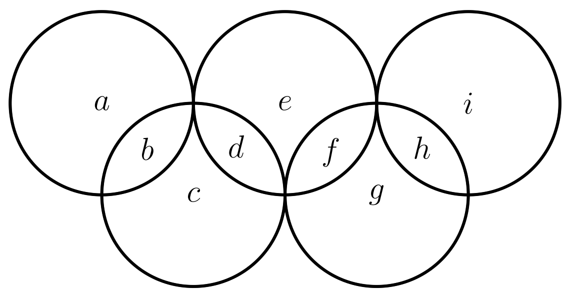
The main problem we want to solve is the following. Consider the set

$$S = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12\}.$$

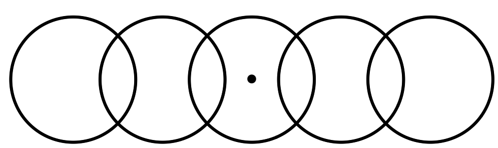
Is it possible to arrange these numbers in the circles so that the sum of the numbers in each line is the same?



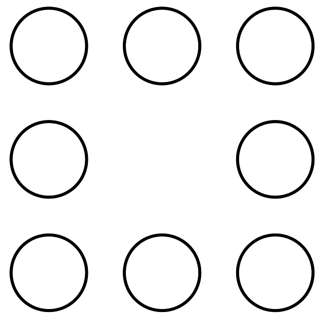
1. Place the number $1, 2, \ldots, 9$ in each of the nine regions (one number in each region) formed by the five circles as shown in the following figure so that the sum of the numbers inside each circle is 14.



2. Place the number $1, 2, \ldots, 9$ in each of the nine regions (one number in each region) formed by the five circles as shown in the following figure so that the sum of the numbers inside each circle is the same. What numbers goes in the region marked by the dot?



3. Consider the array of circles shown below.



(a) Distribute the members of $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ in the circles so that the four sums along horizontal and vertical lines are the same. Let K denote that constant line sum. Call K the *target value*.

(b) Can K be odd?

(c) For what values of K is there a solution?

(d) Among all solutions, what is the largest possible sum of the four corners?

(e) Now let $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Which elements of T can be removed so that the remaining eight element set of digits can be arranged to produce a magic square.

(f) Suppose $R = \{12, 18, 24, 31, 37, 43, 55, 61\}$. Can the members of R be magically distributed? Are there multiple solutions?

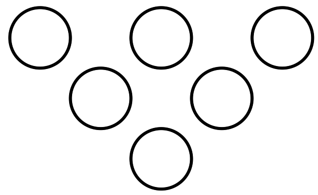
(g) Next suppose you have to make all the sums even or all the sums odd. Using 0 for even and 1 for odd, find all solutions. A good name for an odd-even solution is *pattern*. Find all patterns with an unequal number of 0's and 1's.

(h) Suppose that using all different numbers is not required. Then putting the same number in each position is a solution. Find a way of producing new solutions from old solutions using arithmetic operations. Note that we also need not require that the entries be whole numbers. If the entries can be any real numbers, the solutions have a very nice structure, something you have studied before. Find this structure.

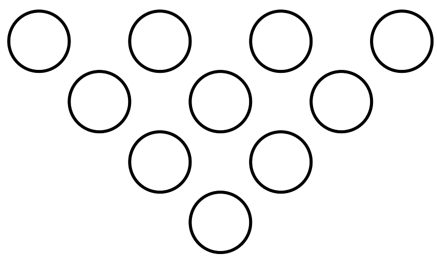
(i) Next we'll consider multiplication instead of addition. Can you distribute the eight members of S so that the product of the numbers along each line is the same. Find a set of eight different positive integers such that the products are the same. What is the smallest positive integer product?

4. The next problem is about magic subtraction triangles.

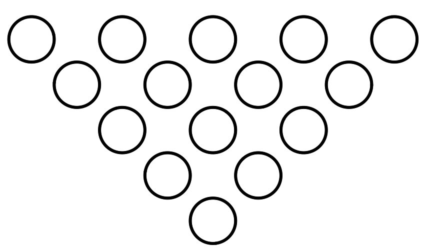
   (a) Put the numbers 1 through 6 in the circles so that each number in the bottom two rows is the positive difference between its two nearest neighbors above it.

   (b) Put the numbers 1 through 10 in the circles so that each number in the bottom three rows is the positive difference between its two nearest neighbors above it.

   (c) Put the numbers 1 through 15 in the circles so that each number in the bottom four rows is the positive difference between its two nearest neighbors above it.

(d) Now try arranging the numbers 1 through 21 in the circles so that each number in the bottom five rows is the positive difference between its two nearest neighbors above it.