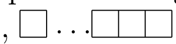
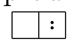
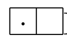
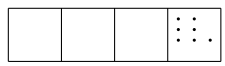
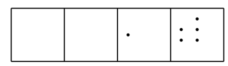
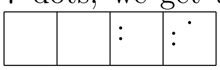
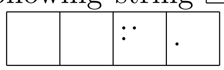
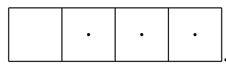
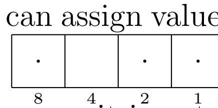
Charlotte Teachers' Institute September 24, 2009

Exploding Dots

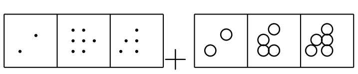
Many thanks to Jim Tanton of St Marks School for the idea of exploding dots. We're going to explore several machines that enable us to represent positive integers is some odd ways. Initially, we're given a tape with empty squares, 
To represent a number n, we put n dots in the rightmost square, and let the machine go to work.

1. The $\boxed{1 \leftarrow 2}$ machine. In this machine, whenever two dots occupy the same square, they are erased (they 'explode') and they are replaced with one dot in the square to their left. Thus  becomes In case we start with 7 dots, we get the following string  $\mapsto$  $\mapsto$  $\mapsto$  $\mapsto$ . Instead of constructing a string of squares and dots, we call this representation 111. As an exercise, see what you get for 19 dots. Also, check to see if the order in which the explosions take place affects the final distribution of dots. Since each dot in a square is worth two dots in the square to its right, we can assign values to each square to what number is represented. For example, , so the value of this dot configuration is $8 + 2 + 1 = 11$. Of course it is not a surprise to us that this is just binary representation.

2. The $\boxed{1 \leftarrow 10}$ machine. In this machine, whenever ten dots occupy the same square, they explode and they are replaced with one dot in the square to their left.

   Exercise. Use the $\boxed{1 \leftarrow 10}$ machine to find the decimal representation of 275.

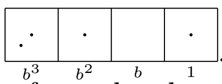
   Let's examine subtraction with the $\boxed{1 \leftarrow 10}$ machine. Consider the problem $275 - 246$. To accomplish this we first devise a notion of negation using *antidots*. We allow two types of symbols in squares, dots .'s and antidots $\circ$. They annihilate each other. Thus, we have

   Can you finish the job?

3. The $\boxed{2 \leftarrow 3}$ machine. In this machine, whenever three dots occupy the same square, they explode and they are replaced with one dot in the square to their left. Let's work out the notation for each of the numbers from 1 to 15.

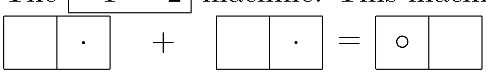
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$R(n)$	1	2	20	21	22	210	211	212	2100	2101	2102	2120	2121	2122	21010

Work you way up to the representation for 24. Notice that the number of digits in the representation jumps as we move to $3, 9$ and 15. Where is the next jump. Why? Is this machine a base-b representation machine for some number b? If so, then . Compute the value of the representation 2101 without help from the chart above.

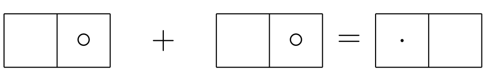
Realizing that each pair of dots in a box is worth three in the next box, we can derive the equations $2b = 3, 2b^2 = 3b, 2b^3 = 3b^2$, etc, all of which give us $b = 3/2$.

Exercise. Find the representation of 123 for this machine. Hint: Solve the equation $\log 1.5^k = \log 123$ to get an idea of how many digits k to use, then find the value of 210...0, where there are k digits.

4. The $\boxed{-1 \leftarrow 2}$ machine. This machine is defined by two equations,



and



Exercises.

(a) Find the $\boxed{-1 \leftarrow 2}$ machine representations for the integers from 1 to 10.

(b) What number is represented by 110110101?

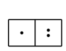
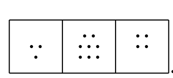
(c) Is this a base b machine like the $\boxed{2 \leftarrow 3}$ machine? If so, find b.

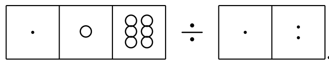
(d) What is the representation of 63? Can you find an algorithm that works for n without find representations of all the numbers 1 through $n - 1$?

5. The $\boxed{1 \leftarrow x}$ machine. In this machine, whenever x dots occupy the same square, they explode and they are replaced with one dot in the square to their left. This leads to polynomial arithmetic. Let's work out polynomial division using the $\boxed{1 \leftarrow x}$ machine. Exercises.

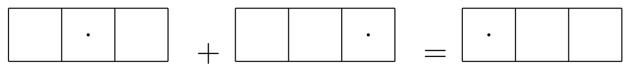
(a) Represent $3x^2 + 8x + 4$ in the $\boxed{1 \leftarrow x}$ machine.

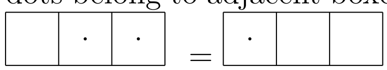
(b) Represent $x + 2$ in the $\boxed{1 \leftarrow x}$ machine.

(c) Now find all instances of  in .

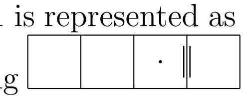
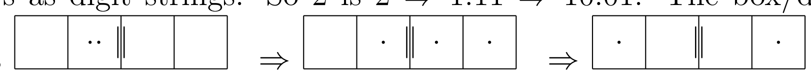
(d) Next try .

For each of the next two problems, 6 and 7, we have the same 'explosion scheme'.

:  In other words, when dots belong to adjacent boxes, they explode to give a dot in the next box over:

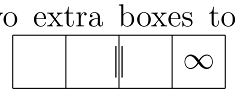


Let's call this the $\boxed{1 \leftarrow 1,1}$ machine.

6. In the first part, we also need a two-way infinite row of boxes. You'll see why we need both directions as we start to count. Of course, 1 is represented as usual, In case we start with 7 dots, we get the following string  The symbol $\|$ represents a special location, which for base b we call a radix point. Instead of using dots in boxes, its more convenient from here on to express integers as digit strings. So 2 is $2 \Rightarrow 1.11 \Rightarrow 10.01$. The box/dot diagram here is  In words, one of the two dots in the first box unexploded producing dots in the two previous boxes. So $2 = 10.01$. Then $3 = 11.01 \Rightarrow 100.01, 4 = 101.01$. Now 5 is tricky: $5 = 102.01 \Rightarrow 101.12 \Rightarrow 101.1111 \Rightarrow 110.0111 \Rightarrow 1000.1001$. Then $6 = 1001.1001 \Rightarrow 1010.0001$.

   (a) Find the representations of the next 5 integers, $7, 8, 9, 10$, and 11.

   (b) Is this a base system in the usual sense. That is, is there a number b for which
   $$6 = b^3 + b + b^{-4}?$$

7. Here's the $\boxed{1 \leftarrow 1,1}$ machine with a black hole. It only takes left infinite strings of boxes with two extra boxes to the right of the radix point, one of which is a black hole:  Here's how this works. Any dots in the box marked ∞ at the end disappear. Otherwise it works just like the machine in question 6. So, $1 = 1$, $2 = 2.00 \Rightarrow 1.11 \Rightarrow 10.01 \Rightarrow 10.00 = 10$, $3 = 11 = 100$, $4 = 12.00 \Rightarrow 11.11 \Rightarrow 100.11 \Rightarrow 101.00$. To find the representation of 5, start with the representation 4, and add one: $5 = 102.00 \Rightarrow 101.11 \Rightarrow 110.01 \Rightarrow 1000.01 \Rightarrow 1000.00 = 1000$.

   (a) Find the representations of the next 5 integers, $7, 8, 9, 10$, and 11.

   (b) Is this a base system in the usual sense. For example, is there a number b for which
   $$4 = b^2 + 1?$$