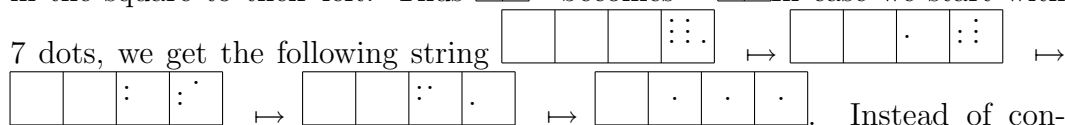


Exploding Dots

We're going to explore several machines that enable us to represent positive integers in some odd ways. Initially, we're given a tape with empty squares. Many thanks to Jim Tanton of St Marks School for the idea of exploding dots. $\square \cdot \square \square \square$
 To represent a number n , we put n dots in the rightmost square, and let the machine go to work.

1. The $\boxed{1 \leftarrow 2}$ machine. In this machine, whenever two dots occupy the same square, they are erased (they 'explode') and they are replaced with one dot in the square to their left. Thus $\square \cdot \square$ becomes $\square \cdot \square$. In case we start with



Instead of constructing a string of squares and dots, we call this representation 111. As an exercise, see what you get for 19 dots. Also, check to see if the order in which the explosions take place affects the final distribution of dots. Since each dot in a square is worth two dots in the square to its right, we can assign values to each square to what number is represented. For example, $\begin{matrix} \cdot & \square & \cdot & \cdot \\ \text{8} & \text{4} & \text{2} & \text{1} \end{matrix}$, so the value of this dot configuration is $8 + 2 + 1 = 11$. Of course it is not a surprise to us that this is just binary representation.

2. The $\boxed{1 \leftarrow 10}$ machine. In this machine, whenever ten dots occupy the same square, they explode and they are replaced with one dot in the square to their left.

Exercise. Use the $\boxed{1 \leftarrow 10}$ machine to find the decimal representation of 275.

Let's examine subtraction with the $\boxed{1 \leftarrow 10}$ machine. Consider the problem $275 - 246$. To accomplish this we first devise a notion of negation using *antidots*. We allow two types of symbols in squares, dots \cdot 's and antidots \circ . They annihilate each other. Thus, we have



Can you finish the job?

3. The $\boxed{2 \leftarrow 3}$ machine. In this machine, whenever three dots occupy the same square, they explode and they are replaced with one dot in the square to their left. Let's work out the notation for each of the numbers from 1 to 15.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$R(n)$	1	2	20	21	22	210	211	212	2100	2101	2102	2120	2121	2122	21010

Work your way up to the representation for 24. Notice that the number of digits in the representation jumps as we move to 3, 9 and 15. Where is the next jump. Why? Is this machine a base- b representation machine for some number b ?

If so, then $\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & & \cdot \\ \hline b^3 & b^2 & b & 1 \\ \hline \end{array}$. Compute the value of the representation 2101 without help from the chart above.

Realizing that each pair of dots in a box is worth three in the next box, we can derive the equations $2b = 3, 2b^2 = 3b, 2b^3 = 3b^2$, etc, all of which give us $b = 3/2$.

Exercise. Find the representation of 123 for this machine. Hint: Solve the equation $\log 1.5^k = \log 123$ to get an idea of how many digits k to use, then find the value of $210\dots 0$, where there are k digits.

4. The $\boxed{-1 \leftarrow 2}$ machine. This machine is defined by two equations,

$$\begin{array}{|c|c|} \hline & \cdot \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \cdot \\ \hline \end{array} = \begin{array}{|c|c|} \hline \circ & \\ \hline \end{array} \quad \text{and}$$

$$\begin{array}{|c|c|} \hline & \circ \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \circ \\ \hline \end{array} = \begin{array}{|c|c|} \hline \cdot & \\ \hline \end{array}$$

Exercises.

- Find the $\boxed{-1 \leftarrow 2}$ machine representations for the integers from 1 to 10.
 - What number is represented by 110110101?
 - Is this a base b machine like the $\boxed{2 \leftarrow 3}$ machine? If so, find b .
 - What is the representation of 63? Can you find an algorithm that works for n without find representations of all the numbers 1 through $n - 1$?
5. The $\boxed{1 \leftarrow x}$ machine. In this machine, whenever x dots occupy the same square, they explode and they are replaced with one dot in the square to their left. This leads to polynomial arithmetic. Let's work out polynomial division using the $\boxed{1 \leftarrow x}$ machine. Exercises.
- Represent $3x^2 + 8x + 4$ in the $\boxed{1 \leftarrow x}$ machine.
 - Represent $x + 2$ in the $\boxed{1 \leftarrow x}$ machine.
 - Now find all instances of $\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}$ in $\begin{array}{|c|c|c|} \hline \cdot & \cdot\cdot & \cdot\cdot\cdot \\ \hline \end{array}$.

(d) Next try

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