Variations on The Perfect Card Trick

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The purpose of this note is to explore the so-called Perfect Card Trick and its relatives using a marriage theorem not as well-known as Hall's Marriage Theorem. We start with the two marriage theorems.

The Marriage Theorem. Suppose you're given a set A of women and a set B of men, with at least as many men as women. Each woman has a set of requirements and each man has a set of attributes. So, some men satisfy a woman's requirements and some don't. Thus, for each woman x in A, there is a subset $E(x)$ of men that satisfy x's requirements. Our job is to marry off all the women to men who satisfy their requirements. There are two important theorems and also two important data structures that help us to carry out the task. First the structures.

Matrix representation. We can build a *boolean matrix* (using just 0 and 1 as entries) to depict the data. The rows of the matrix are the women and the columns are the men. We put a 1 in a row and column if the column(man) satisfies the requirements of the row(woman). A solution to the problem is another boolean matrix of the same size where each row has exactly one 1 and each column has at most one 1.

Bipartite graph model. A *graph* is a set of *vertices* (also called nodes), some pairs of which are joined by *edges* (also called branches). The degree of a vertex is the number of edges that are incident to it. Note that in the 'crown' graph below, the degree of each vertex in A is 3. In our problem, let's suppose there are $|A| = n$ women and $|B| = m$ men, where $n \leq m$. Then the graph has $n + m$ vertices. The edges join women with men who satisfy their requirements. Thus there are no edges joining pairs of women or pairs of men. That is why we call this a *bipartite graph.* A solution to the problem thus become a subset of the edge set with the property that each vertex of A has degree one and each vertex of B that is joined by a member of the edge set has degree one also.

Now for the theorems that help solve the problem. One theorem that guarantees the result and shows how to do it is called Hall's Marriage Theorem. It give necessary and sufficient conditions for the construction of a one-to-one function f from A into B such that for all x, $f(x)$ satisfies the requirements of x. In other words, x should marry $f(x)$. **Hall's Marriage Theorem**. Suppose A and B satisfy the following. For each k, $1 \leq k \leq n$ and each subset C of A with k members, the set of all y in B that satisfy the requirements of at least one x in C has at least k members. If we denote by $E(x)$ the set of men that satisfy x's requirements, then we are saying $|E(x)| \geq 1$; $|E(x) \cup E(y)| \geq 2$ for all $x \neq y$; $\ldots$; $| \cup_{i=1}^{k} E(x_i)| \geq k$ for all $k$element subsets of A. Under these conditions, such a one-to-one function exists. Next we turn to **Holshouser's Marriage Theorem**. It is not as well-known as Hall's theorem, but it gets the job done in the card problem. Again we are give the two sets A and B and the sets $E(x)$ of elements of B that satisfy x's requirements. Then there exists a one-to-one function f from A into B such that for all x, $f(x)$ satisfies the requirements of x if the following two conditions are satisfied:

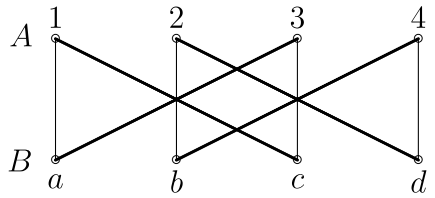
1. $|E(x)| = |E(y)|$ for all x, y in A and
2. Any two men are eligible to marry the same number of women.

Here's an example of Holshouser's Theorem. Suppose a 52-card deck of playing cards is divided into 13 piles of size 4. Then it is possible to pick a different denomination(value) from each pile. Think of the women as the denominations and the men as the 4-card piles. A pile is eligible if it has that denomination in it. So in the graph model, each denomination x has four edges. In the matrix model, the rows are the denominations and the piles are the columns. The 13×13 matrix has 52 1's. The solution is a matrix with exactly one 1 in each row. Notice that in this case we are allowing multiple edges between some pairs of vertices. The marriage theorems still hold. We simple have to could the multiple branches in computing the vertex degrees.

The card trick that we just showed you is based on a theorem in graph theory that is called the Marriage Theorem. The issue is whether the elements of a finite set A can be matched up with the elements of a set B subject to certain conditions. The theory works whether the sets are equal or unequal. So I will consider both cases.

In the first example, A and B are equal. We have drawn *branches* (or edges) from each member of A to some of the members of B. The members of A and B are called *nodes* (or vertices). Notice that each node on the graph has the same number of branches touching it. This number is called the *degree* of the node. Here each node has degree 2. We will call this condition the marriage hypothesis for equal sets. Anytime that the marriage hypothesis is met then we will be able to pair off 1 to 1 the members of A with the members of B by using some of these branches. Thus, if we call A the women and call B the men and the marriage hypothesis is met, then we can marry off the women and the men into pairs by using some of these branches.

In our drawing as an example we might pair 1 to c, 2 to b, 3 to a, and 4 to d.

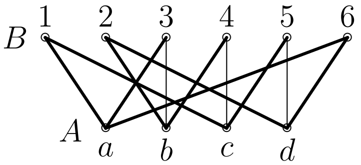


An example of a bipartite graph

Let me repeat. This pairing can always be done when B and A are equal and the number of branches touching each node is the same for all nodes.

The case where one set (call it B) is bigger than the other set (call it A) is illustrated below.

Note that each node in B has the same number of branches (2) touching it. Also, each node in A has the same number of branches (3) touching it. This condition is called the marriage hypothesis for unequal sets. The marriage theorem says that you can marry (one to one) the members of the smaller set (i.e. A) with a subset of the members of the larger set (i.e. B). Of course, since B is bigger than A you will have some members of B who are not married. Thus, in our illustration, you might marry a to 6, b to 2, c to 1, and d to 5.



A bipartite graph with partite sets of size 4 and 6 satisfying Holshouser's Theorem.

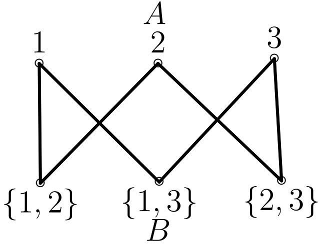
The Card Trick. There are n cards in a deck numbered 1, 2, 3,$\cdots$, n. Someone in the audience, say Cal, draws m cards where $m < n$ and gives the m cards to Bill. Bill looks at the m cards and then lays down $m-1$ of the cards in some order and the order is important. From this information his partner Amy can tell the audience that the mth card is.

The marriage theorem shows that this trick can always be worked if $n \leq m! + m - 1$. Thus, if $m = 5$ then $n \leq 5! + 5 - 1 = 124$. If $m = 4$ then $n \leq 4! + 4 - 1 = 27$. If $m = 3$ then $n \leq 3! + 3 - 1 = 8$. If $m = 2$ then $n \leq 2! + 2 - 1 = 3$.

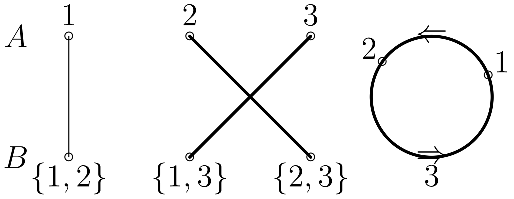
We now apply this theory with $n = 3, m = 2$ to show how we came up with the simple trick that we started with.

Now there are three combinations of 2 from the three numbers $1, 2, 3$ namely $\{1, 2\}, \{1, 3\}, \{2, 3\}$. We now let these 3 doubleton sets be the 3 members of set B. Also, the single numbers $1, 2, 3$ are the 3 members of set A.

Note that A and B are the same size with 3 members each. We draw branches from A to B as shown below. Note the pattern here.



Of course, each node of A and each node of B has 2 branches connected to it. So we can marry the members of A one to one with the members of B. One way to do this is as follows. This is easy to remember, Let us now play our card game again. Suppose the audience draws cards $\{2, 3\}$. I then tell my partner the number 2 and he knows that 3 is the other member of the set $\{2, 3\}$.



The tables below show the relationship among the three parameters n, the deck size, m the sample size, and h the number of cards that can be hidden.

n	m	h
8	5	2

We now illustrate the theory for the following cases.

Problem 1. Let the deck have $n = 6$ cards and the sample $m = 4$ cards. Now $6 < 4! + 4 - 1 = 27$. So the trick is easy to work out since 6 is much smaller than 27. Let us call the 6 cards $1, 2, 3, 4, 5, 6$. Since $m = 4$, the victim will draw 4 cards.

Then the master will arrange 3 cards in some order for his partner, who will then tell the audience the 4th card. There are $C_4^6 = 15$ combinations of 4 cards which we now list in the following way which we can also call alphabetical order. Also, we can call each combination of 4 cards a word. There are also $6 \cdot 5 \cdot 4 = 120$ permutations of three which is much larger than 15.

1. $\{1, 2, 3, 4\}$ $(1, 2, 3)$
2. $\{1, 2, 3, 5\}$ $(1, 2, 5)$
3. $\{1, 2, 3, 6\}$
4. $\{1, 2, 4, 5\}$
5. $\{1, 2, 4, 6\}$
6. $\{1, 2, 5, 6\}$
7 $\{1, 3, 4, 5\}$
8. $\{1, 3, 4, 6\}$
9. $\{1, 3, 5, 6\}$
10. $\{1, 4, 5, 6\}$
11. $\{2, 3, 4, 5\}$
12. $\{2, 3, 4, 6\}$
13. $\{2, 3, 5, 6\}$
14. $\{2, 4, 5, 6\}$
15. $\{3, 4, 5, 6\}$

We wish to assign to each of these 15 words $\{a, b, c, d\}$, where $a < b < c < d$, a permutation of three (x, y, z) where $x \neq y, x \neq z, y \neq z$ and $x, y, z \in \{a, b, c, d\}$. It is not necessary that $x < y < z$. We want to do this in such a way that each of the 15 words is assigned to a different permutation of three. This means that if the permutation of three (x, y, z) is assigned to the word (a, b, c, d), then (x, y, z) will identify the word for us.

As an example, consider the first word $(1, 2, 3, 4)$. We can assign to this word $(1, 2, 3, 4)$ any of the following 24 permutations of three, $(1, 2, 3)$, $(1, 3, 2)$, $(1, 2, 4)$, $(1, 4, 2)$, $(1, 3, 4)$, $(1, 4, 3)$, $(2, 1, 3)$, $(2, 3, 1)$, $(2, 1, 4)$, $(2, 4, 1)$, $(2, 3, 4)$, $(2, 4, 3)$, $(3, 1, 2)$, $(3, 2, 1)$, $(3, 1, 4)$, $(3, 4, 1)$, $(3, 2, 4)$, $(3, 4, 2)$, $(4, 1, 2)$, $(4, 2, 1)$, $(4, 1, 3)$, $(4, 3, 1)$ $(4, 2, 3)$, $(4, 3, 2)$.

Suppose we assign to $\{1, 2, 3, 4\}$ the permutation $(4, 1, 3)$. This will mean that if the 4 cards drawn are 1, 2, 3, 4 and I lay down (in order) the 3 cards 4, 1, 3, then my partner will know that the 4th card must be 2 since $(4, 1, 3)$ is associated with the set $(1, 2, 3, 4)$.

Now 6 is much smaller than $4! + 4 - 1 = 27$. This is also the reason why 15 is much smaller than 120. Since 15 is much smaller than 120, this problem is very easy to solve. Also, we have started you off.

After you have solved this problem, we will want to get together again and let somebody illustrate his/her solution. This will make sure that everyone understand the trick.

Solution:

$(1, 2, 3, 4) - (1, 2, 3)$
$(1, 2, 3, 5) - (1, 2, 5)$
$(1, 2, 3, 6) - (1, 2, 6)$
$(1, 2, 4, 5) - (2, 4, 5)$
$(1, 2, 4, 6) - (4, 2, 1)$
$(1, 2, 5, 6) - (6, 2, 1)$
$(1, 3, 4, 5) - (1, 3, 4)$
$(1, 3, 4, 6) - (1, 3, 6)$
$(1, 3, 5, 6) - (1, 3, 5)$
$(1, 4, 5, 6) - (6, 5, 4)$
$(2, 3, 4, 5) - (5, 4, 3)$
$(2, 3, 4, 6) - (6, 4, 2)$
$(2, 3, 5, 6) - (6, 5, 2)$
$(2, 4, 5, 6) - (5, 4, 6)$
$(3, 4, 5, 6) - (3, 6, 4)$

Problem 2. Let $n = 7$ and $m = 3$.

Now $7 < 3! + 3 - 1 = 8$. So the problem can be solved. However, since 7 is close to 8, this problem is going to be a good bit harder than Problem 1. Now there are $C_3^7 = \binom{7}{3} = 35$ combinations of 3, taken from a set of 7 cards. Also, there are $7 \cdot 6 = 42$ ordered pairs (x, y) from 7 cards where $x \neq y$ and $x, y \in \{1, 2, 3, 4, 5, 6, 7\}$. Note that $35 < 42$. The reason we can solve the problem is that there are more ordered pairs than there are combinations of 3. The marriage theorem tells us a solution exists. However, you will find a solution ad hoc. The following two lists show how all 35 combinations of 3 and all 42 ordered pairs. The marriage theorem tells us that the first list can be marched (one to one) with 35 members of the second list.

Using ad hoc methods you can match the 35 members of the first list one to one with 35 members of the second list in any way that you wish. We suggest that you go down the first list and try to match each word with the first member in the second list that is available. We have started you off.

1.	$\{1,2,3\}$	$(1,2)$	19.	$\{2,3,7\}$
2.	$\{1,2,4\}$	$(1,4)$	20.	$\{2,4,5\}$
3.	$\{1,2,5\}$	$(1,5)$	21.	$\{2,4,6\}$
4.	$\{1,2,6\}$	$(1,6)$	22.	$\{2,4,7\}$
5.	$\{1,2,7\}$		23.	$\{2,5,6\}$
6.	$\{1,3,4\}$		24.	$\{2,5,7\}$
7.	$\{1,3,5\}$		25.	$\{2,6,7\}$
8.	$\{1,3,6\}$		26.	$\{3,4,5\}$
9.	$\{1,3,7\}$		27,	$\{3,4,6\}$
10.	$\{1,4,5\}$		28.	$\{3,4,7\}$
11.	$\{1,4,6\}$		29.	$\{3,5,6\}$
12.	$\{1,4,7\}$		30.	$\{3,5,7\}$
13.	$\{1,5,6\}$		31.	$\{3,6,7\}$
14.	$\{1,5,7\}$		32.	$\{4,5,6\}$
15.	$\{1,6,7\}$		33.	$\{4,5,7\}$
16.	$\{2,3,4\}$		34.	$\{4,6,7\}$
17	$\{2,3,5\}$		35.	$\{5,6,7\}$
18.	$\{2,3,6\}$			

1.	$(1,2)$	15.	$(3,4)$	29.	$(5,6)$
2.	$(1,3)$	16.	$(3,5)$	30.	$(5,7)$
3.	$(1,4)$	17.	$(3,6)$	31.	$(6,1)$
4.	$(1,5)$	18.	$(3,7)$	32.	$(6,2)$
5.	$(1,6)$	19.	$(4,1)$	33.	$(6,3)$
6.	$(1,7)$	20.	$(4,2)$	34.	$(6,4)$
7.	$(2,1)$	21.	$(4,3)$	35.	$(6,5)$
8.	$(2,3)$	22.	$(4,5)$	36.	$(6,7)$
9.	$(2,4)$	23.	$(4,6)$	37.	$(7,1)$
10.	$(2,5)$	24.	$(4,7)$	38.	$(7,2)$
11.	$(2,6)$	25.	$(5,1)$	39.	$(7,3)$
12.	$(2,7)$	26.	$(5,2)$	40.	$(7,4)$
13.	$(3,1)$	27.	$(5,3)$	41.	$(7,5)$
14.	$(3,2)$	28.	$(5,4)$	42.	$(7,6)$

When you get through with Problem 2, you can play the trick at your tables until you get tired of it. However, we would like to watch you and check your solution.

Solution: The solution in table form is the following

	1	2	3	4	5	6	7
1		3	4	2	2	2	2
2			4	5	3	3	3
3	5			5	6	1	1
4	5	6	6		6	1	1
5	6	6	7	7		7	1
6	7	7	7	**7**			
7		4	4		2		

To see how to interpret this table, consider the bold 7 in row 6. If the three card set Bill gets is $\{4, 6, 7\}$, Bill hides the 7 and presents Amy the ordered pair $(6, 4)$. See the bold entries in the next table.

In a list form, the solution is the following.

$\{1,2,3\}$	$(1,2)$	$\{2,3,7\}$	$(2,7)$
$\{1,2,4\}$	$(1,4)$	$\{2,4,5\}$	$(2,4)$
$\{1,2,5\}$	$(1,5)$	$\{2,4,6\}$	$(4,2)$
$\{1,2,6\}$	$(1,6)$	$\{2,4,7\}$	$(7,2)$
$\{1,2,7\}$	$(1,7)$	$\{2,5,6\}$	$(5,2)$
$\{1,3,4\}$	$(1,3)$	$\{2,5,7\}$	$(7,5)$
$\{1,3,5\}$	$(3,1)$	$\{2,6,7\}$	$(6,2)$
$\{1,3,6\}$	$(3,6)$	$\{3,4,5\}$	$(3,4)$
$\{1,3,7\}$	$(3,7)$	$\{3,4,6\}$	$(4,3)$
$\{1,4,5\}$	$(4,1)$	$\{3,4,7\}$	$(7,3)$
$\{1,4,6\}$	$(4,6)$	$\{3,5,6\}$	$(3,5)$
$\{1,4,7\}$	$(4,7)$	$\{3,5,7\}$	$(5,3)$
$\{1,5,6\}$	$(5,1)$	$\{3,6,7\}$	$(6,3)$
$\{1,5,7\}$	$(5,7)$	$\{4,5,6\}$	$(4,5)$
$\{1,6,7\}$	$(6,1)$	$\{4,5,7\}$	$(5,4)$
$\{2,3,4\}$	$(2,3)$	**{4,6,7}**	**(6,4)**
$\{2,3,5\}$	$(2,5)$	$\{5,6,7\}$	$(5,6)$
$\{2,3,6\}$	$(2,6)$		

Problem 3. Let $n = 8$ and $m = 3$.

Now $8 = 3! + 3 - 1$. So this problem can be solved. However, the solution will be hard since we have taken the trick to the theoretical limit for $m = 3$. You basically do what you did for Problem 3. However, at the end you will have to juggle things around a bit. In the interest of time, we will give you the solution that we have worked out. You can use either the 2 lists or the compact table.

	1	2	3	4	5	6	7	8
1		3	4	2	2	2	2	4
2	8		4	5	7	3	3	3
3	5	5		5	8	1	1	1
4	5	6	6		6	1	1	6
5	6	6	7	7		3	1	1
6	7	7	7	7	8		8	2
7	8	4	4	8	8	5		2
8	6	4	4	5	2	3	3	

The two lists are the same except they are backwards. The table corresponds to the two lists. Suppose the 3 cards are $(1, 3, 8)$. From the first list, the set $(1, 3, 8)$ goes with $(3, 8)$. If I lay down $(3, 8)$ then from the second list I know that $(3, 8)$ goes with $(1, 3, 8)$. To use the compact table note that 3 is read in the left column and 8 is read in the top row. The intersection point is 1.

$(1,2,3)$	$(1,2)$	$(2,4,7)$	$(7,2)$
$(1,2,4)$	$(1,4)$	$(2,4,8)$	$(8,2)$
$(1,2,5)$	$(1,5)$	$(2,5,6)$	$(5,2)$
$(1,2,6)$	$(1,6)$	$(2,5,7)$	$(2,5)$
$(1,2,7)$	$(1,7)$	$(2,5,8)$	$(8,5)$
$(1,2,8)$	$(2,1)$	$(2,6,7)$	$(6,2)$
$(1,3,4)$	$(1,3)$	$(2,6,8)$	$(6,8)$
$(1,3,5)$	$(3,1)$	$(2,7,8)$	$(7,8)$
$(1,3,6)$	$(3,6)$	$(3,4,5)$	$(3,4)$
$(1,3,7)$	$(3,7)$	$(3,4,6)$	$(4,3)$
$(1,3,8)$	$(3,8)$	$(3,4,7)$	$(7,3)$
$(1,4,5)$	$(4,1)$	$(3,4,8)$	$(8,3)$
$(1,4,6)$	$(4,6)$	$(3,5,6)$	$(5,6)$
$(1,4,7)$	$(4,7)$	$(3,5,7)$	$(5,3)$
$(1,4,8)$	$(1,8)$	$(3,5,8)$	$(3,5)$
$(1,5,6)$	$(5,1)$	$(3,6,7)$	$(6,3)$
$(1,5,7)$	$(5,7)$	$(3,6,8)$	$(8,6)$
$(1,5,8)$	$(5,8)$	$(3,7,8)$	$(8,7)$
$(1,6,7)$	$(6,1)$	$(4,5,6)$	$(4,5)$
$(1,6,8)$	$(8,1)$	$(4,5,7)$	$(5,4)$
$(1,7,8)$	$(7,1)$	$(4,5,8)$	$(8,4)$
$(2,3,4)$	$(2,3)$	$(4,6,7)$	$(6,4)$
$(2,3,5)$	$(3,2)$	$(4,6,8)$	$(4,8)$
$(2,3,6)$	$(2,6)$	$(4,7,8)$	$(7,4)$
$(2,3,7)$	$(2,7)$	$(5,6,7)$	$(7,6)$
$(2,3,8)$	$(2,8)$	$(5,6,8)$	$(6,5)$
$(2,4,5)$	$(2,4)$	$(5,7,8)$	$(7,5)$
$(2,4,6)$	$(4,2)$	$(6,7,8)$	$(6,7)$

(1, 2)	(1, 2, 3)	(4, 1)	(1, 4, 5)	(7, 1)	(1, 7, 8)
(1, 3)	(1, 3, 4)	(4, 2)	(2, 4, 6)	(7, 2)	(2, 4, 7)
(1, 4)	(1, 2, 4)	(4, 3)	(3, 4, 6)	(7, 3)	(3, 4, 7)
(1, 5)	(1, 2, 5)	(4, 5)	(4, 5, 6)	(7, 4)	(4, 7, 8)
(1, 6)	(1, 2, 6)	(4, 6)	(1, 4, 6)	(7, 5)	(5, 7, 8)
(1, 7)	(1, 2, 7)	(4, 7)	(1, 4, 7)	(7, 6)	(5, 6, 7)
(1, 8)	(1, 4, 8)	(4, 8)	(4, 6, 8,)	(7, 8)	(2, 7, 8)
(2, 1)	(1, 2, 8)	(5, 1)	(1, 5, 6)	(8, 1)	(1, 6, 8)
(2, 3)	(2, 3, 4)	(5.2)	(2, 5, 6)	(8, 2)	(2, 4, 8)
(2, 4)	(2, 4, 5)	(5, 3)	(3, 5, 7)	(8, 3)	(3, 4, 8)
(2, 5)	(2, 5, 7)	(5, 4)	(4, 5, 7)	(8, 4)	(4, 5, 8)
(2, 6)	(2, 3, 6)	(5, 6)	(3, 5, 6)	(8, 5)	(2, 5, 8)
(2, 7)	(2, 3, 7)	(5, 7)	(1, 5, 7)	(8, 6)	(3, 6, 8)
(2, 8)	(2, 3, 8)	(5, 8)	(1, 5, 8)	(8, 7)	(3, 7, 8)
(3, 1)	(1, 3, 5)	(6, 1)	(1, 6, 7)		
(3, 2)	(2, 3, 5)	(6, 2)	(2, 6, 7)		
(3, 4)	(3, 4, 5)	(6, 3)	(3, 6, 7)		
(3, 5)	(3, 5, 8)	(6, 4)	(4, 6, 7)		
(3, 6)	(1, 3, 6)	(6, 5)	(5, 6, 8)		
(3, 7)	(1, 3, 7)	(6, 7)	(6, 7, 8)		
(3, 8)	(1, 3, 8)	(6, 8)	(2, 6, 8)		

To illustrate the case $n = 8, m = 5, h = 3$, consider the table of five element subsets and their ordered pair counterparts.

$\{1,2,3,4,5\}$	$(1,2)$
$\{1,2,3,4,6\}$	$(1,3)$
$\{1,2,3,4,7\}$	$(1,4)$
$\{1,2,3,4,8\}$	$(1,8)$
$\{1,2,3,5,6\}$	$(1,5)$
$\{1,2,3,5,7\}$	$(1,7)$
$\{1,2,3,5,8\}$	$(2,1)$
$\{1,2,3,6,7\}$	$(1,6)$
$\{1,2,3,6,8\}$	$(2,3)$
$\{1,2,3,7,8\}$	$(2,7)$
$\{1,2,4,5,6\}$	$(2,6)$
$\{1,2,4,5,7\}$	$(2,5)$
$\{1,2,4,5,8\}$	$(2,4)$
$\{1,2,4,6,7\}$	$(4,1)$
$\{1,2,4,6,8\}$	$(2,8)$
$\{1,2,4,7,8\}$	$(4,2)$
$\{1,2,5,6,7\}$	$(5,1)$
$\{1,2,5,6,8\}$	$(5,2)$
$\{1,2,5,7,8\}$	$(5,7)$
$\{1,2,6,7,8\}$	$(6,1)$
$\{1,3,4,5,6\}$	$(3,4)$
$\{1,3,4,5,7\}$	$(3,1)$
$\{1,3,4,5,8\}$	$(3,5)$
$\{1,3,4,6,7\}$	$(3,6)$
$\{1,3,4,6,8\}$	$(3,8)$
$\{1,3,4,7,8\}$	$(3,7)$
$\{1,3,5,6,7\}$	$(5,3)$
$\{1,3,5,6,8\}$	$(5,6)$

$\{1,3,5,7,8\}$	$(5,8)$
$\{1,3,6,7,8\}$	$(6,3)$
$\{1,4,5,6,7\}$	$(4,5)$
$\{1,4,5,6,8\}$	$(4,6)$
$\{1,4,5,7,8\}$	$(7,1)$
$\{1,4,6,7,8\}$	$(8,1)$
$\{1,5,6,7,8\}$	$(6,7)$
$\{2,3,4,5,6\}$	$(4,3)$
$\{2,3,4,5,7\}$	$(3,2)$
$\{2,3,4,5,8\}$	$(5,4)$
$\{2,3,4,6,7\}$	$(6,2)$
$\{2,3,4,6,8\}$	$(6,4)$
$\{2,3,4,7,8\}$	$(7,2)$
$\{2,3,5,6,7\}$	$(6,5)$
$\{2,3,5,6,8\}$	$(6,8)$
$\{2,3,5,7,8\}$	$(7,3)$
$\{2,3,6,7,8\}$	$(7,6)$
$\{2,4,5,6,7\}$	$(7,4)$
$\{2,4,5,6,8\}$	$(8,2)$
$\{2,4,5,7,8\}$	$(7,5)$
$\{2,4,6,7,8\}$	$(7,8)$
$\{2,5,6,7,8\}$	$(8,5)$
$\{3,4,5,6,7\}$	$(4,7)$
$\{3,4,5,6,8\}$	$(8,3)$
$\{3,4,5,7,8\}$	$(8,4)$
$\{3,4,6,7,8\}$	$(8,6)$
$\{3,5,6,7,8\}$	$(8,7)$
$\{4,5,6,7,8\}$	$(4,8)$

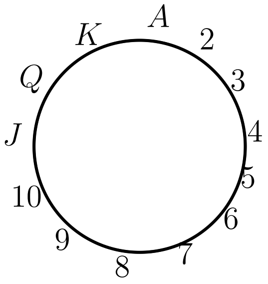
Next we demonstrate and explain to you a commercial card trick using the regular 52 card deck. He will use the sample size $m = 5$. This means that 5 cards will be drawn and $h = 1$ card will be hidden.

Note that $n = 52 < 5! + 5 - 1 = 124$. Since 52 is much smaller than 124, a computer could do much better than this. However, this commercial trick is very easy to remember. You can then play with this trick until it is time for you to leave.

Let D be a set of n cards. Typically the cards are numbered from 1 to n, but they could be a deck of ordinary playing cards with $n = 52$. Amy and Bill work together to baffle Cal as follows. Cal picks a sample of m cards from the deck. Then Bill selects one (or two) cards from the sample and arranges the rest in order. He hides the selected card(s) and gives Amy the rest in the order he has selected. She looks at the list and announces the hidden card(s). This problem set poses problems about the relationship between the numbers m and n. For example, given m we can ask, what is the largest n for which the trick can be done.

What if there are two hidden cards? The answer with one hidden card is $n = m! + m - 1$, so, for example, when three cards are in the sample $n = 3! + 3 - 1 = 8$. We can view the selection of the card to hide as a binary operation $*$ on the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If the three sample cards are x, y, and z, then exactly one of $x * y, x * z, y * x, y * z, z * x, z * y$ belongs to the set $\{x, y, z\}$. Suppose $x * y = z$. This means Bill gives Amy the two cards x and y in that order, and Amy knows that the hidden card is $x * y = z$.

One version of the trick is well-known. If D is an ordinary deck of 52 playing cards and $m = 5$, there is a cute algorithm for picking out the hidden card from each sample. Here's how it goes. Among any five cards, there must be at least two cards of the same suit. Counting clockwise around in a circle, one of these two cards is at most 6 larger than the other. Think of the 13 cards of that suit as the numbers from 1 to 13, with Ace equal 1, Jack as 11, etc. Recall that each set of three cards can be arranged in six different orders. If we designate an ordering on the cards themselves, this means that we can define six numbers. Equate $lmh, lhm, mlh, mhl, hlm, hml$ with the numbers 1 through 6 respectively where l means the least of the three values, m the middle one, and h the highest value. Now among the five cards you are given, find two of the same suit. Why can you always do this? If these are cards U and V, locate U and V on the circle with the numbers $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$ written clockwise around the circle. Now one of the cards U or V is within six clockwise steps of the other. Choose this as the *hidden card*. The other becomes the *anchor card*. Then figure the clockwise 'distance' from the anchor card to the hidden card. Finally arrange the three remaining cards in the order that determines the number of clockwise steps to take to get from the first card to the hidden card. For each set of five cards listed below, tell which one you'd hide and put the other four in the right order, with the anchor first.



For example, if the five cards are $2♣, 3♢, 6♡, 8♡, J♠$, you would hide the $8♡$ and arrange the other four cards as follows: $6♡, 2♣, J♠, 3♢$. The anchor card $6♡$ tell

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Amy that the hidden card is a heart in the range $7, 8, 9, 10, J = 11, Q = 12$ The final three cards are in the order lhm which corresponds to 2 steps, so we take two steps from the $6\heartsuit$ to get to the $8\heartsuit$. What do you do when the three cards you have to order have two or more on the same value? You put those of the same value in alphabetical order, so $2\clubsuit < 2\diamondsuit < 2\heartsuit < 2\spadesuit$.

Problem 4. For each set of five cards listed below, decide which one you would hide, which one you'd use as an anchor card and how you would arrange the other three.

1. $K\clubsuit, 3\diamondsuit, 6\heartsuit, 8\heartsuit, J\spadesuit$

   Solution: Hide $8\heartsuit$ and order the other four $6\heartsuit, 3\diamondsuit, K\clubsuit, J\spadesuit$

2. $7\clubsuit, 7\diamondsuit, 7\heartsuit, J\diamondsuit, 7\spadesuit$

   Solution: Hide $J\diamondsuit$ and order the other four $7\diamondsuit, 7\heartsuit, 7\spadesuit, 7\clubsuit$

3. $J\clubsuit, Q\diamondsuit, K\heartsuit, A\spadesuit, 7\spadesuit$

   Solution: Hide $7\spadesuit$ and order the other four $, A\spadesuit, K\heartsuit, Q\diamondsuit, J\clubsuit$

4. $5\clubsuit, 8\diamondsuit, 10\spadesuit, Q\heartsuit, K\clubsuit$

   Solution: Hide $5\clubsuit$ and order the other four $K\clubsuit, Q\heartsuit, 8\diamondsuit, 10\spadesuit$

5. $2\diamondsuit, 5\diamondsuit, 6\heartsuit, 8\spadesuit, J\clubsuit$

   Solution: Hide $5\diamondsuit$ and order the other four $2\diamondsuit, 8\spadesuit, 6\heartsuit, J\clubsuit$

6. $J\clubsuit, 9\diamondsuit, 6\heartsuit, Q\heartsuit, J\spadesuit$

   Solution: Hide $Q\heartsuit$ and order the other four $6\heartsuit, J\spadesuit, J\clubsuit, 9\diamondsuit$

7. $2\clubsuit, 8\diamondsuit, 8\heartsuit, 8\spadesuit, J\spadesuit$

   Solution: Hide $J\spadesuit$ and order the other four $, 8\spadesuit, 8\diamondsuit, 2\clubsuit, 8\heartsuit$

Problem 5. Let $m = 6$. Find the largest integer n for which your group can build an algorithm to determine the hidden card from the deck $D = \{1, 2, 3, \ldots, n\}$. Of course n is at least 52 because you could use the algorithm above to solve the problem. Try to find a solution with n at least 200.

Problem 6. Let $m = 5$. Find the largest integer n for which your group can build an algorithm to determine **two** hidden cards from the deck $D = \{1, 2, 3, \ldots, n\}$. Then choose an n much smaller than this and devise an algorithm that you can actually use to find the two hidden cards.