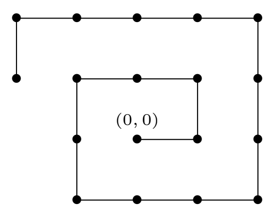
Charlotte Teachers Institute
Bug Problems

All the following problems except number 2 refer to a bijective function $f : \mathbb{Z}^+ \to \mathbb{Z} \times \mathbb{Z}$ such that $d(f(n), f(n+1)) \leq \sqrt{2}$ for all n and $f(0) = (0,0)$. Such a function is called a *crawl*. Here, $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$ is the set of nonnegative integers, and $\mathbb{Z}$ is the set of all integers.

1. A bug crawls around the plane at a uniform rate, one unit per minute. He starts at the origin at time 0 and crawls one unit to the right, arriving at $(1,0)$, turns $90°$ left and crawls another unit to $(1,1)$, turns $90°$ left again, and crawls two units. He continues to make $90°$ left turns as shown in the figure. Let $g(t)$ denote the position in the plane after t minutes, where t is an integer. Thus, for example, $g(0) = (0,0)$, $g(6) = (-1,-1)$, and $g(16) = (-2,2)$. (The path of the bug establishes a one-to-one correspondence between the non-negative integers and the integer lattice points of the plane.)



   (a) Where is the bug after exactly 2008 minutes? How many turns has the bug made after 2008 minutes?

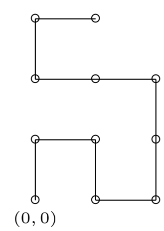
   (b) How many minutes does it take for the bug to get to the ordered pair $(19, 99)$?

   (c) Does there exist an integer t such that $g(t)$ and $g(t+23)$ are exactly 17 units apart? If so, find the smallest such t.

   (d) Now suppose one bug leaves exactly 99 minutes after the other. What is the closest they ever get to each other? In other words, what is the smallest possible value of $D(g(t), g(t+99)), t \geq 0$, where D represents the distance function? What is the smallest value of t for which this distance is achieved.

(e) Now suppose one bug leaves exactly 100 minutes after the other. What is the closest they ever get to each other? In other words, what is the smallest possible value of $D(t(n), g(t+100)), t \geq 0$, where D represents the distance function? What is the smallest value of t for which this distance is achieved.

2. This time the bug stays in the first quadrant. It crawls from the origin up one unit after one minute, over to $(1, 1)$ after another minute, and down to $(1, 0)$ after three minutes. Each time it gets to an axis, it makes a right turn away from the origin, crawls one unit, then makes another right turn, staying in the first quadrant.

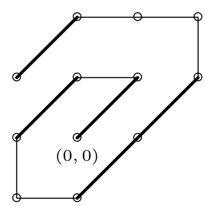


   (a) Where is the bug after exactly 2008 minutes?

   (b) How many minutes does it take for the bug to get to the ordered pair $(19, 99)$?

3. A bug starts from the origin on the plane and crawls one unit upwards to $(0, 1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2, 1)$. Then during the third minute, it crawls three units upward, arriving at $(2, 4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl n units during minute n and then makes a $90°$ turn either left or right. The bug continues this until after 16 minutes, it finds itself back at the origin. Its path does not intersect itself. What is the smallest possible area of the 16-gon traced out by its path?

4. This time our bug can make turns other than right turns. Consider the path shown below, again starting at the origin. Each unit segment between lattice points take exactly one minute to traverse and each diagonal segment of length $\sqrt{2}$ also takes one minute.



   (a) Where is the bug after exactly 2008 minutes?

   (b) How many minutes does it take for the bug to get to the ordered pair $(19, 99)$?

   (c) Find the smallest t for which the bugs position $g(t)$ after t minutes is at least 100 units away from the origin. For this value of t, how many turns has the bug made?

5. This time our bug moves around based on where it is. Let

$$A = \{(m, n) \mid m > -2/3 \text{ and } -m - 1 \leq n < 2m + 1\},$$

$$B = \{(m, n) \mid n \geq 2m + 1 \text{ and } n \geq m/2\},$$

and

$$C = \{(m, n) \mid n \leq m/2 \text{ and } n \leq m - 1\}.$$

As before $g(t)$ is the position of the bug at time t. Now we are given that $g(0) = (0, 0)$. Also, when the bug belongs to set A at time t, it moves up one unit at $t + 1$. If $g(t) = (a, b)$ belongs to B, then $g(t + 1) = (a - 1, b - 1)$, and if $g(t) = (a, b)$ belongs to C, then $g(t + 1) = (a - 1, b)$.

(a) Where is the bug after exactly 1000 minutes?

(b) How many minutes does it take for the bug to get to the ordered pair $(21, -22)$?

(c) Find the smallest t for which the bugs position $g(t)$ after t minutes is at least 100 units away from the origin. For this value of t, how many turns has the bug made?