Tchebyshev posets

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Dedicated to Louis Billera on his sixtieth birthday

Abstract

We construct for each n an Eulerian partially ordered set T_n of rank $n+1$ whose ce-index provides a non-commutative generalization of the n-th Tchebyshev polynomial. We show that the order complex of each T_n is shellable, homeomorphic to a sphere, and that its face numbers minimize the expression $\max_{|x|\leq 1}\left|\sum_{j=0}^{n}(f_{j-1}/f_{n-1})\cdot 2^{-j}\cdot (x-1)^j\right|$ among the f-vectors of all $(n-1)$-dimensional simplicial complexes. The duals of the posets constructed have a recursive structure similar to face lattices of simplices or cubes, offering the study of a new special class of Eulerian partially ordered sets to test the validity of Stanley's conjecture on the non-negativity of the cd-index of all Gorenstein* posets.

Introduction

The object of this paper is the study of a sequence $T_0, T_1, T_2, \ldots$ of Eulerian partially ordered sets constructed in such a way that for each n, the ce-index of T_n is a non-commutative generalization of the Tchebyshev polynomial $T_n(x)$. We call them Tchebyshev posets. One remarkable property of this sequence of posets is "self-similarity": for each atom a of T_{n+1} the partially ordered set of elements above a is isomorphic to a copy of T_n. Since each Tchebyshev poset may be represented as the face poset of a CW-complex that is homeomorphic to a ball, at the level of CW-complexes we may state that any face of a "dual Tchebyshev cell" is a "dual Tchebyshev cell", in analogy to the statements "every face of a simplex is a simplex" and "every face of a cube is a cube". Therefore the same way one studies simplicial complexes and cubical complexes, it is possible to focus on CW-complexes whose faces are dual Tchebyshev cells.

The construction that yields our Tchebyshev posets may be defined in greater generality. This possibility is explored in section 2. Here we define an operator T that assigns a partially ordered set

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$T(Q)$ to any given partially ordered set Q, by introducing a partial order on its non-singleton intervals. An interval of Q is considered smaller than another in $T(Q)$ if either this holds element-wise, or the smaller interval is an initial segment of the larger one. The operator is such that a rank function of Q induces a rank function of $T(Q)$. The operator T does not assign a graded poset to a graded poset, but if Q has a unique atom covering a unique minimum element, then $T(Q)$ has a unique minimum element. Moreover, given a locally finite poset Q that has a unique minimum element x_0 covered by a unique atom y_0, every interval of $T(Q)$ is Eulerian if and only if the same holds for $Q \setminus \{x_0\}$. Hence we may obtain a new, more complex Eulerian poset from any given one by adding a new minimum element below its minimum element, and applying the operator T to it. The partially ordered sets T_n we call Tchebyshev posets are obtained by applying this procedure to the simplest possible Eulerian posets, i.e., the "ladder" posets which have exactly 2 elements at any given rank between the maximum and the minimum element. It is a worthy subject of future research, what is the effect of the operator T on other, more complicated (and not even necessarily Eulerian) posets.

In section 3 we take a closer look at the partially ordered set T_n and describe an important labeling of its cover relations, using only four symbols. This labeling is used to present a shelling of the order complex of T_n in section 5.

In section 2 we describe the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ as a triangulation of the boundary of the n-dimensional cross-polytope. This description implies that the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ is homeomorphic to a sphere. The triangulation may be realized with "straight" faces, which raises the open question whether the order complex would also have a line shelling induced by a geometric picture. It is also worth noting that the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ provides a balanced triangulation of the boundary complex of the cross-polytope that needs less faces than barycentric subdivision.

The shelling of the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ presented in section 5 allows to describe the *cd*-index of T_n in section 7 in terms of an "ascent-descent" statistic, in a way that is analogous to Purtill's work [19] on the *cd*-index of the Boolean algebra. The major difference is that while in the case of the Boolean algebra one needs to sum over permutations, here we sum essentially over all strings of given length formed by 4 letters. The computation of the *cd*-index is followed by formulas for the *ce*-index and the flag f-vector in section 8. Finally, in section 9 we use a well-known extremal property of the Tchebyshev polynomials to show an extremal property of the f-vector of the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ among f-vectors all simplicial complexes of the same dimension.

The most interesting possible use of Tchebyshev posets is the verification of Stanley's conjecture on the non-negativity of the *cd*-index of Gorenstein* posets in a new special setting that is genuinely different from the (partially) known simplicial and cubical cases. An outline of this proposed future research is presented in section 10.

1 Preliminaries

1.1 Graded partially ordered sets

A partially ordered set P is *locally finite* if every interval $[x, y] \subseteq P$ contains a finite number of elements. An element $y \in P$ *covers* $x \in P$ if $y > x$ and there is no element between x and y. We will use the notation $y \succ x$. A function $\rho : P \longrightarrow \mathbb{Z}$ is a rank function for P if $\rho(y) = \rho(x) + 1$ is satisfied whenever y covers x. A partially ordered set may have more than one rank function, but the restriction of any rank function to any interval $[x, y] \subseteq P$ is unique up to a constant shift. Therefore the *rank* $\rho(x, y)$ *of an interval* $[x, y]$, defined by $\rho(x, y) = \rho(y) - \rho(x)$ is the same number for any rank function, and it is equal to the common length of all maximal chains connecting x and y. We say that a finite partially ordered set is *graded* when it has a unique minimum element $\widehat{0}$, a unique maximum element $\widehat{1}$, and a rank function ρ. In this situation we usually require $\rho(\widehat{0}) = 0$ and we call $\rho(\widehat{1})$ the rank of the graded partially ordered set. Given a graded partially ordered set P of rank $n + 1$ and a set $S \subseteq \{1, \ldots, n\}$ we define the *S–rank selected subposet of P* to be the poset $P_S = \{x \in P \ : \ \rho(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}$. We denote by $f_S(P)$ the number of maximal chains of P_S. Equivalently, $f_S(P)$ is the number of chains $x_1 < \cdots < x_{|S|}$ in P such that $\{\rho(x_1), \ldots, \rho(x_{|S|})\} = S$. The vector $(f_S(P) \ : \ S \subseteq \{1, \ldots, n\})$ is called the *flag f-vector* of P.

Two equivalent encodings of the flag f-vector are the *flag h-vector* $(h_S(P) \ : \ S \subseteq \{1, \ldots, n\})$ and the *flag ℓ-vector* $(\ell_S(P) \ : \ S \subseteq \{1, \ldots, n\})$. They are defined by $h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P)$ and $\ell_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} (-1)^{|T|} \cdot f_T(P)$.

1.2 Eulerian posets

A graded partially ordered set is *Eulerian* if every interval $[x, y]$ of positive rank in it satisfies $\sum_{x \leq z \leq y} (-1)^{\rho(z)} = 0$. All linear relations holding for the flag f-vector of an arbitrary Eulerian poset of rank n were determined by Bayer and Billera in [2]. These linear relations were rephrased by J. Fine as follows (see the paper [5] by Bayer and Klapper). For any $S \subseteq \{1, \ldots, n\}$ define the non-commutative monomial $u_S = u_1 \ldots u_n$ by setting

$$u_i = \begin{cases} b & \text{if } i \in S, \\ a & \text{if } i \notin S \end{cases}$$

Then the polynomial $\Psi_{ab}(P) = \sum_S h_S u_S$ in non-commuting variables a and b, called the *ab-index of P*, is a polynomial of $c = a + b$ and $d = ab + ba$. This form of $\Psi_{ab}(P)$ is called the *cd-index of P*. Further proofs of the existence of the cd-index may be found in [24], in [4], and in [13]. It was noted by Stanley in [24] that the existence of the cd-index is equivalent to saying that the ab-index rewritten as a polynomial of $c = a + b$ and $e = a - b$ involves only even powers of e. It was observed by Bayer and Hetyei in [3] that the coefficients of the resulting *ce-index* may be computed using a formula that is analogous to the definition of the ℓ-vector. In fact, given a ce-word $u_1 \cdots u_n$, let S be the set of positions i satisfying $u_i = e$. Then the coefficient $L_S(P)$ of the ce-word is given by $L_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} \left(-\frac{1}{2}\right)^{|T|} f_T(P)$. The fact that the ce-index is a polynomial of c and e^2 is equivalent to stating that $L_S(P) = 0$ unless S is an *even* set, that is, a union of disjoint intervals of even cardinality.

Following [22] we call a partially ordered set P *lower Eulerian* if it has a unique minimum element $\widehat{0}$, and for every $t \in P$ the interval $[0, t]$ is Eulerian.

1.3 Chain-edge labelings

Given a graded poset P of rank $n+1$, let $\mathcal{M}(P)$ be the set of maximal chains of P, and let $\mathcal{E}(P) = \{(x, y) : x, y \in P, x \prec y\}$ be the set of cover relations of P. An *edge labeling* of P is a map $\lambda : \mathcal{E}(P) \longrightarrow \Lambda$, into some partially ordered set of labels Λ. A *chain-edge labeling* of P is a map $\lambda : \mathcal{M}(P) \times \mathcal{E}(P) \longrightarrow \Lambda$, into some poset Λ satisfying the following axiom:

(CE) If two maximal chains $C : \widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \widehat{1}$ and $C' : \widehat{0} = x'_0 \prec x'_1 \prec \cdots \prec x'_n \prec x'_{n+1} = \widehat{1}$ satisfy $x_i = x'_i$ for $i = 1, 2, \ldots, k$, then we have $\lambda(C, (x_{i-1}, x_i)) = \lambda(C', (x'_{i-1}, x'_i))$ for $i = 1, 2, \ldots, k$.

In other words, for a chain-edge labeling $\lambda : \mathcal{M}\mathcal{E}(P) \longrightarrow \Lambda$ the value of $\lambda(C, (x, y))$ depends only on x, y and the chain $C(\rho(x)) : \widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_{\rho(x)-1} \prec x_{\rho(x)}$, which is a maximal chain of the graded poset $[\widehat{0}, x]$. We call an interval $[x, y]$ enriched by maximal chain r in $[\widehat{0}, x]$ a *rooted interval*, and denote it by $[x, y]_r$. Every edge labeling λ induces a chain-edge labeling λ' defined by $\lambda'(C, (x, y)) = \lambda(x, y)$.

A chain-edge labeling $\lambda : \mathcal{M}(P) \times \mathcal{E}(P) \longrightarrow \Lambda$ *has the first atom property* (or it is an *FA-labeling*) if in every rooted interval $[x, y]_r$ there is a unique atom a such that

$$\lambda_r(C, (x, a)) < \lambda_r(C', (x, a'))$$

for every other atom a' of $[x, y]$ and every pair (C, C') of maximal chains of $[x, y]_r$, containing a and a' respectively. We call the atom a the *first atom* of the rooted interval $[x, y]_r$ with respect to the chain-edge labeling. FA-labelings were introduced by Billera and Hetyei in [7]. Every graded partially ordered set has an FA-labeling. In fact, drawing the poset in the plane and numbering all cover relations $x \prec y$ covering x from left to right with $1, 2, \ldots$, yields an FA-labeling $\lambda : \mathcal{M}(P) \times \mathcal{E}(P) \longrightarrow \mathbb{N}$ that does not depend on $\mathcal{M}(P)$, i.e., it is an edge labeling.

Given an FA-labeling, let ϕ be the operation which assigns to every rooted interval $[x, y]_r$ its first atom. Let us fix a maximal chain $C : \widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \widehat{1}$. Consider the function

$\psi_C : [1,n] \longrightarrow [1,n]$ defined by $\psi_C(i) = \max\{j \ : \ x_i = \phi\left([x_{i-1}, x_j]_{C(i-1)}\right\}$. (Here $C(i-1)$ is the saturated chain $C(i-1) : \widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_{i-1}$ in $[\widehat{0}, x_{i-1}]$.) Using the function ψ_C we may associate a family of intervals $\mathcal{I}_C = \{[i, \psi(C,i)] \ : \ i \in [1,n], \quad \psi(C,i) \neq n+1\}$ to each maximal chain. Let us enumerate the maximal chains $C_1, \ldots, C_m$ of P in an order that extends the lexicographic order on their labels. Then a subset $\{x_s \ : \ s \in S\}$ occurs first in $C : \widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \widehat{1}$ if and only if $S \subseteq [1,n]$ *blocks* $\mathcal{I}_C$, that is every $I \in \mathcal{I}_C$ has a nonempty intersection with S. This is stated in [7, Section 2], as a generalization of the proof of [6, Theorem 2.1].

Intuitively speaking, the meaning of $[i,j] \in \mathcal{I}_C$ is that there exists a chain $C' = \widehat{0} \prec x_1 \cdots \prec x_{i-1} \prec x'_i \prec \cdots \prec x'_j \prec x_{j+1} \prec \cdots \prec x_{n+1} = \widehat{1}$ that precedes C' in the enumeration, and provides an "earlier" path from x_{i-1} to x_{j+1} in the Hasse diagram of P. Hence every partial chain, not included in the union of the preceding maximal chains, must contain at least one element whose rank is between i and j.

CR-labelings are defined as chain-edge labelings for which every rooted interval has a unique maximal chain whose labels are rising. If this chain is always the first in the lexicographic order, then it is called a *CL-labeling*. It was shown in [7] that CL-labelings are FA-labelings, and that for such labelings the minimal elements in every family of intervals $\mathcal{I}_C$ are singletons. Introducing h_S for the number of maximal chains satisfying $\mathcal{I}_C = \{\{s\} \ : \ s \in S\}$, we obtain the flag h-vector. A CR- or CL-labeling that comes from an edge-labeling (and is not depending on the maximal chain) is called ER- or EL-labeling, respectively.

All terms introduced here may be found either in the paper of Björner and Wachs [9] or in the work of Billera and Hetyei in [7]. CL-labelings were originally introduced by Björner and Wachs in [8], ER-labelings are called R-labelings in Stanley's book [23, Section 3.13].

1.4 Simplicial complexes

A simplicial complex $\triangle$ is a family of subsets of a vertex set V such that every singleton $\{v\}$ (where $v \in V$) belongs to $\triangle$, and $\triangle$ is closed under taking subsets. The elements of $\triangle$ are called *faces* and the *dimension* of a face $\sigma \in \triangle$ is defined by $\dim(\sigma) = |\sigma| - 1$. The maximal faces are called *facets*. The number of i-dimensional faces is usually denoted by f_i, and the vector $(f_{-1}, f_0, \ldots, f_d)$ is often called the *f-vector* of the simplicial complex.

In this paper we will only be concerned with *order complexes* of partially ordered sets. Given a partially ordered set P, the vertices of its order complex $\triangle(P)$ are the elements of P, and the faces are the increasing chains. In other words, a subset $\{x_1, \ldots, x_k\}$ of P is defined to be a face if and only if $x_1 < \cdots < x_k$. When the partially ordered set P is graded then $\triangle(P)$ is a *cone* over its maximum element $\widehat{1}$ and its minimum element $\widehat{0}$, that is $\sigma \subseteq P$ is a face of the order complex if and only if any of $\sigma \setminus \{\widehat{0}\}$, $\sigma \setminus \{\widehat{1}\}$, $\sigma \cup \{\widehat{0}\}$, or $\sigma \cup \{\widehat{1}\}$ is a face. Because of this, many papers in the literature mean by the order complex of a graded poset P the order complex of $P \setminus \{\widehat{0}, \widehat{1}\}$. In this paper we will always have to indicate precisely which of the minimum or maximum elements (if any) should be omitted, since this will depend on the geometric situation.

A simplicial complex is *pure*, if every facet has the same dimension. A pure simplicial $\triangle$ complex is *shellable* if there is an enumeration $F_0, F_1, \ldots, F_t$ of its facets such that for every $k \in \{2, \ldots, t\}$ the collection of faces of F_k contained in some earlier F_i is a pure simplicial complex of dimension $\dim \triangle - 1$. Equivalently, there exists a unique minimal face $R(F_k)$ of F_k such that every face $\sigma \subseteq F_k$ not contained in any earlier F_i contains $R(F_k)$.

Given a graded partially ordered set P of rank $n+1$, the order complex of $P \setminus \{\widehat{0}, \widehat{1}\}$ is pure of dimension $n-1$.

Lemma 1.1 *If an FA-labeling of a graded poset P satisfies that for every saturated chain C the minimal elements of the interval system $\mathcal{I}_C$ are singletons $\mathcal{I}_C = \{\{s\} : s \in S\}$, then the enumeration of the saturated chain in any order that extends the lexicographic order on their labels yields a shelling of the order complex of $P \setminus \{0, \widehat{1}\}$.*

This lemma is straightforward, since it follows directly from the definitions that for every facet C of the order complex, $R(C)$ is the set of those elements whose rank s satisfies $\{s\} \in \mathcal{I}_C$.

In particular, as noted in [7], every CL-labeling is an FA-labeling with the above property, and thus yields a shelling of the order complex.

2 Elementary properties of general Tchebyshev posets

The elementary properties of Tchebyshev posets are most easily proven by introducing a definition that is more general than what is needed for most of this paper.

Definition 2.1 *Given a locally finite partially ordered set Q, we define the* Tchebyshev poset *$T(Q)$ of Q as follows. Its elements are all ordered pairs $(x, y) \in Q \times Q$ satisfying $x < y$, and we set $(x_1, y_1) \leq (x_2, y_2)$ when $y_1 \leq x_2$ or $x_1 = x_2$ and $y_1 \leq y_2$.*

In other words, the elements of $T(Q)$ are identifiable with all intervals of Q that are not singletons. We consider an interval larger than the other if either every element of the larger interval is larger than every element of the smaller interval or the smaller interval is an "initial segment" of the larger interval.

Remark 2.2 We extracted this definition from the study of the special Tchebyshev posets T_n whose properties will be explored in depth in this paper. Without the focus on those posets, the question naturally arises, would including singletons allow to create similarly interesting partially ordered sets?

Proposition 2.3 *$T(Q)$ is a partially ordered set.*

Proof: The relation we defined is obviously reflexive. Before proving antisymmetry, let us note that $(x_1, y_1) \leq (x_2, y_2)$ implies $y_1 \leq y_2$. Thus $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_1, y_1)$ implies $y_1 = y_2$, making $y_1 \leq x_2$ impossible, since $x_2 < y_2$. Therefore the only way to satisfy $(x_1, y_1) \leq (x_2, y_2)$ is by $x_1 = x_2$ and the relation defined is antisymmetric. Assume $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_3, y_3)$. Then $y_1 \leq y_2$ and $y_2 \leq y_3$ imply $y_1 \leq y_3$. By $(x_2, y_2) \leq (x_3, y_3)$ either $x_3 \geq y_2$ or $x_3 = x_2$ holds. In the first case we have $x_3 \geq y_2 \geq y_1$ in the second we have either $x_3 = x_2 \geq y_1$ or $x_3 = x_2 = x_1$ by $(x_1, y_1) \leq (x_2, y_2)$. Therefore the relation defined is transitive. $\diamond$

Proposition 2.4 *If $\rho : Q \longrightarrow \mathbb{Z}$ is a rank function for Q then setting $\rho(x, y) = \rho(y)$ provides a rank function for $T(Q)$. In fact, the set of elements covering $(x, y) \in T(Q)$ is*

$$\{(x, \dot{y}) \,:\, y \prec \dot{y}\} \cup \{(y, \dot{y}) \,:\, y \prec \dot{y}\}.$$

(Here $\dot{y}$ denotes an arbitrary element covering y in Q.)

Proof: Consider a cover relation $(x_1, y_1) \prec (x_2, y_2)$ in $T(Q)$. By definition we must have $y_1 < y_2$ and either $x_2 \geq y_1$ or $x_2 = x_1$. If $x_2 > y_1$ then (y_1, x_2) is strictly between (x_1, y_1) and (x_2, y_2) and we get a contradiction. Hence we have $x_2 \in \{x_1, y_1\}$. We show that in either case y_2 must cover y_1 in Q. Given a y strictly between y_1 and y_2 we have $(x_1, y_1) < (x_1, y) < (x_2, y_2)$ if $x_2 = x_1$, and $(x_1, y_1) < (y_1, y) < (x_2, y_2)$ if $x_2 = y_1$. This concludes the proof of the fact that only the relations listed in the statement may be cover relations. On the other hand, given a cover relation $y \prec \dot{y}$, the relations $(x, y) \prec (x, \dot{y})$ and $(x, y) \prec (y, \dot{y})$ are cover relations, since any (x_3, y_3) strictly between (x, y) and $(x, \dot{y})$ or $(y, \dot{y})$ must satisfy $y < y_3 < \dot{y}$, which is not possible. $\diamond$

Since only those elements of Q occur in an ordered pair $(x, y) \in T(Q)$ that are comparable to at least one other element, we may remove from Q all those elements which are incomparable to all others, without changing $T(Q)$.

Proposition 2.5 *Assume that every element of Q is comparable to at least one other element of Q. Then $T(Q)$ has a unique minimum element if and only if Q has a unique minimum element x_0 covered by a unique atom y_0. In that case the unique minimum element of $T(Q)$ is (x_0, y_0).*

Proof: The proof of the "if" part is straightforward: if x_0 is the unique minimum element of Q and only y_0 covers x_0 then any $(x, y) \in T(Q)$ satisfies either $x = x_0$ or $x \geq y_0$.

Assume now that (x_0, y_0) is the unique minimum element of $T(Q)$. We claim that $x_0 \leq x$ holds for all $x \in Q$. If x is not a maximum element of Q then there exists an $(x, y) \in T(Q)$ for some $y \in Q$. By $(x_0, y_0) \leq (x, y)$, we must either have $x \geq y_0 \geq x_0$ or $x = x_0$. In either case $x \geq x_0$ holds. If $x \in Q$ is a maximal element of Q then there is at least one $z \in Q$ to which x is comparable, and this x being maximal we must have $z < x$. Since z is not maximal, we already know $x_0 \leq z$, implying $x_0 < x$. Therefore x_0 is the unique minimum element of Q. Observe next that y_0 must cover x_0, since any y strictly between x_0 and y_0 would give rise to $(x_0, y) < (x_0, y_0)$. To conclude the proof observe that no other atom z covers x_0, since such an atom would give rise to an (x_0, z) that is incomparable with (x_0, y_0). ◇

Lemma 2.6 *Given $x_1 < y_1 \leq y_2 \in Q$, the interval $[(x_1, y_1), (x_1, y_2)] \subseteq T(Q)$ is isomorphic to $[y_1, y_2] \subseteq Q$.*

In fact, an element $(x, y) \in T(Q)$ belongs to $[(x_1, y_1), (x_1, y_2)]$ if and only if $x = x_1$ and $y \in [y_1, y_2]$.

Proposition 2.7 *Assume that every element of Q is comparable to some other element and that $T(Q)$ has a unique minimum element (x_0, y_0). Then every interval of $T(Q)$ is an Eulerian poset if and only the same holds for every interval of $Q \setminus \{x_0\}$.*

Proof: The necessity is a consequence of Lemma 2.6, since for any $[y_1, y_2] \subseteq Q \setminus \{x_0\}$ we have $x_0 < y_1 \leq y_2$, thus (x_0, y_1) and (x_0, y_2) are the endpoints of an interval in $T(Q)$ that is isomorphic to $[y_1, y_2]$.

To prove sufficiency, again by Lemma 2.6, we may restrict our attention to those $[(x_1, y_1), (x_2, y_2)] \subseteq T(Q)$ which satisfy $x_1 \neq x_2$, and verify that for such intervals

$$\sum_{(x,y)\in[(x_1,y_1),(x_2,y_2)]} (-1)^{\rho(x,y)} = 0$$

holds. Recall that, by Proposition 2.4, every (x, y) in the above sum satisfies $\rho(x, y) = \rho(y)$ where $\rho(y)$ is the rank of y in Q, hence we need to show

$$\sum_{(x,y)\in[(x_1,y_1),(x_2,y_2)]} (-1)^{\rho(y)} = 0$$

Every element $(x, y) \in [(x_1, y_1), (x_2, y_2)]$ satisfies exactly one of $x = x_1$, $x_1 < x < x_2$, or $x = x_2$. Depending on this condition, we may partition our summands $(-1)^{\rho(x,y)}$ into three partial sums as follows:

(i) If $x = x_1$ then (x, y) belongs to $[(x_1, y_1), (x_2, y_2)]$ if and only if $y_1 \leq y \leq x_2$. The contribution of the summands in this first subset is $\sum_{y_1 \leq y \leq x_2} (-1)^{\rho(y)}$. Since every interval of Q is Eulerian, we obtain a contribution of $(-1)^{\rho(x_1)} \cdot \delta_{y_1,x_2}$, where δ is the Kronecker delta function.

(ii) If $x_1 < x < x_2$ then x must satisfy $y_1 \leq x < x_2$ and y must satisfy $x < y \leq x_2$. The contribution of the summands falling into this category is

$$\sum_{y_1 \leq x < x_2} \sum_{x < y \leq x_2} (-1)^{\rho(y)} = \sum_{y_1 \leq x < x_2} (-1)^{\rho(x)+1} = (-1)^{\rho(x_2)} \cdot (1 - \delta_{y_1,x_2})$$

(iii) If $x = x_2$ then y must satisfy $x_2 < y \leq y_2$. The contribution of the summands in this last subset is $\sum_{x_2 < y \leq y_2} (-1)^{\rho(y)} = (-1)^{\rho(x_2)+1}$.

Summing up the three partial sums yields 0. ◇

3 The Tchebyshev posets T_n

Consider the partially ordered set $\mathbb{P}^{\pm}$ with the elements

$$-1 < 1 < -2, 2 < -3, 3 < -4, 4 < \cdots$$

Pairs of elements separated with a comma are considered incomparable. This partially ordered set has a unique minimum element -1, which is covered by the unique atom 1, hence, by Proposition 2.5 the Tchebyshev poset $T(\mathbb{P}^{\pm})$ has a unique minimum element $(-1, 1)$. It is easy to verify that

$$\rho(x) = \begin{cases} |x| - 1 & \text{if } x \neq -1, \\ -1 & \text{if } x = -1 \end{cases}$$

is a rank function for this poset, so Proposition 2.4 implies that $T(\mathbb{P}^{\pm})$ is a graded poset with rank function $\rho(x, y) = |y| - 1$. Every interval $(x, y) \subset \mathbb{P}^{\pm} \setminus \{-1\}$ has exactly two elements at each rank strictly between $\rho(x)$ and $\rho(y)$, from which fact it is easy to conclude every interval of $\mathbb{P}^{\pm}$ is Eulerian. Therefore, by Proposition 2.7, every interval of $T(\mathbb{P}^{\pm})$ is Eulerian.

We are now in the position to introduce the Tchebyshev posets T_n for each natural number n.

Definition 3.1 *The Tchebyshev poset T_n is the interval $[(-1, 1), -(n+1), -(n+2))]$ in the generalized Tchebyshev poset $T(\mathbb{P}^{\pm})$.*

Figure 1 represents the Tchebyshev poset T_3. The meaning of the labels on the edges will be explained in section 5. Let us summarize what we already know about the Tchebyshev poset T_n as a consequence

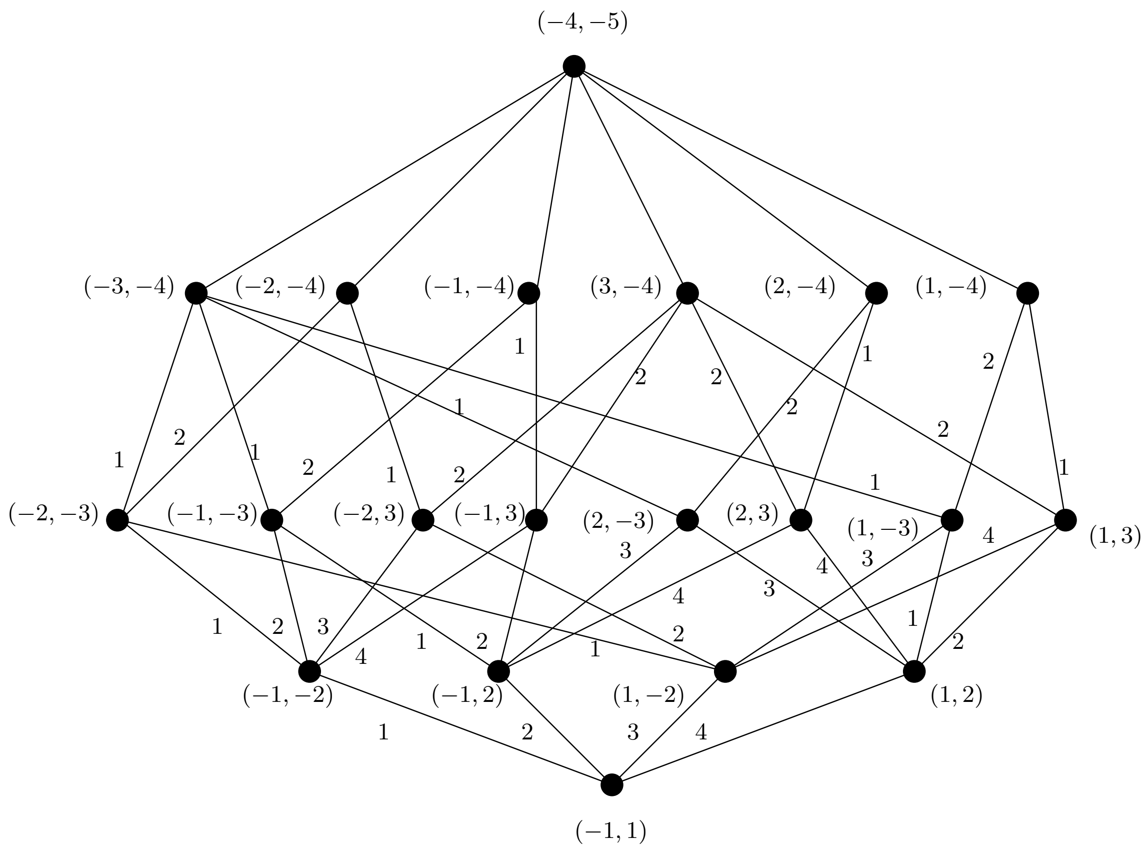

Figure 1: The Tchebyshev poset T_3

of its definition and of our general theory. It has a unique minimum element $\widehat{0} = (-1, 1)$, and a unique maximum element $\widehat{1} = (-(n+1), -(n+2))$, it is graded and Eulerian. The elements strictly between the minimum and the maximum element are pairs of nonzero integers (x, y) such that $|x| < |y|$, $|x|, |y| \in \{1, 2, \ldots, n\}$, with the restriction that for $|y| = n + 1$ we must have $y = n + 1$. The relation $(x_1, y_1) < (x_2, y_2)$ holds if $|y_1| < |y_2|$ and at least one of the following is satisfied:

(i) $|x_1| < |x_2|$ or
(ii) $y_1 = x_2$ or
(iii) $x_1 = x_2$.

The rank of the element (x, y) is $|y| - 1$. In particular, the rank of T_n is $n + 1$. The cover relations of T_n are the following:

(i) $\widehat{0} = (-1, 1) \prec (\varepsilon 1, \eta 2)$,
(ii) $(x, y) \prec (x, \varepsilon(|y| + 1))$ for $|x| < |y| < n$,
(iii) $(x, y) \prec (y, \eta(|y| + 1))$ for $|x| < |y| < n$, and

(iv) $(\varepsilon x, -(n+1)) \prec (-(n+1), -(n+2)) = \widehat{1}$.

Here ε and η are signs from the set $\{-,+\}$. In particular, every element of rank less than n is covered by exactly four elements. The cover relations may be naturally labeled by the symbols L^+, L^-, R^+, R^- as follows. We think of each label as an operator assigning the covering element to the covered element.

Definition 3.2 *Given $\varepsilon, \eta \in \{+,-\}$ and (x,y) with $1 \leq |x| < |y| \leq n$ we set*

$$R^{\varepsilon}(x,y) = (x, \varepsilon(|y|+1)) \quad \text{and} \quad L^{\eta}(x,y) = (y, \eta(|y|+1)). \tag{1}$$

In other words the, operators R^{ε} remove the right component, the operators L^{η} remove the left component of (x,y). The sign ε resp. η indicates the sign of the new letter $(|y|+1)$ introduced.

Using (1), we extend the definition of R^{ε} and L^{η} to $\widehat{0} = (-1,1)$. We extend only L^- and R^- to elements of the form $(x, \pm n)$. We extend only L^- to the coatoms $(x, -(n+1))$ which are sent into $\widehat{1} = (-(n+1), -(n+2))$ by L^-.

In proving the results of section 5, the following lemmata will be instrumental.

Lemma 3.3 *The range of the operators, L^+, L^-, R^+, and R^- is pairwise disjoint. In fact, $(x,y) \neq \widehat{0}$ is the image of some L^{η} exactly when $|y| = |x|+1$, and it is the image of some R^{ε} exactly when $|y| > |x|+1$. Moreover, the sign of any operator whose image is (x,y) is the same as the sign of y.*

This lemma is straightforward.

Lemma 3.4 *For any $x, y \in \mathbb{P}^{\pm}$ satisfying $|x| < |y|$, we have $(x', y') < (x,y)$ if and only if $(x', y') < (x, -y)$.*

In fact, if (x,y) is the image of some element under R^+ then $(x,-y)$ is the image of the same element under R^- and vice versa. The analogous claim also holds for the operators L^+ and L^-, so the set of elements covered by (x,y) and $(x,-y)$ is the same.

Lemma 3.5 *If (x,y) belongs to the image of $L^{\mathrm{sign}(y)}$, then $(x,y) = L^{\mathrm{sign}(y)} L^{\mathrm{sign}(x)}(x', y')$ is satisfied for any (x', y') of rank $\rho(x,y) - 2$.*

In fact, the given assumptions imply $|x| = |y| - 1$, and

$$L^{\mathrm{sign}(y)} L^{\mathrm{sign}(x)}(x', y') = L^{\mathrm{sign}(y)}(y', x) = (x, y).$$

Lemma 3.6 *For all* $(x, y) \in T_n$ *of rank at most* $(n-1)$ *the identities* $L^{\eta}R^{\varepsilon}(x,y) = L^{\eta}L^{\varepsilon}(x,y)$, $R^{\varepsilon}L^{+}(x,y) = R^{\varepsilon}L^{-}(x,y)$, *and* $R^{\varepsilon}R^{+}(x,y) = R^{\varepsilon}R^{-}(x,y)$ *hold.*

The verification of this lemma is a straightforward substitution into the definitions.

In analogy to Boolean algebras and the face lattice to the cube the duals of the Tchebyshev posets are "self-similar" in the following sense.

Proposition 3.7 *For any* $n \geq 1$, *and any atom* $(\varepsilon 1, \eta 2) \in T_n$ *the interval* $[(\varepsilon 1, \eta 2), \widehat{1}]$ *is isomorphic to* T_{n-1}.

Proof: Without loss of generality we may assume that the atom in question is $(-1, -2)$. An element $(\varepsilon i, \eta j)$ is in $[(-1,-2), \widehat{1}]$ if and only if $i > 2$ or $\varepsilon = -$. In other words, exactly those pairs $(\varepsilon i, \eta j)$ are above $(-1,-2)$ for which the letters 1 or 2 occur only with negative sign. Hence the elements of the interval $[(-1,-2), \widehat{1}] = [(-1,-2), (-n, -(n+1))]$ are open intervals of the partially ordered set

$$-1 < -2 < -3, 3 < -4, 4 < \cdots < -n, n < -(n+1) < -(n+2)$$

ordered by the Tchebyshev relations. Consider the map ϕ from the partially ordered set

$$-1 < -2 < -3, 3 < -4, 4 < \cdots < -n, n < -(n+1) < -(n+2)$$

onto the partially ordered set

$$-1 < 1 < -2, 2 < -3, 3 < \cdots < -(n-1), (n-1) < -n < -(n+1)$$

given by the formula

$$\phi(\varepsilon i) = \begin{cases} \varepsilon(i-1) & \text{if } i > 2 \\ 1 & \text{if } i = 2 \\ -1 & \text{if } i = 1 \end{cases}$$

This map is a bijection, its inverse is given by

$$\phi^{-1}(\varepsilon i) = \begin{cases} \varepsilon(i+1) & \text{if } i \geq 2 \\ -2 & \text{if } i = 1, \text{ and } \varepsilon = 1 \\ -1 & \text{if } i = 1, \text{ and } \varepsilon = -1. \end{cases}$$

It is easy to verify that ϕ is an isomorphism of partially ordered sets, which induces an isomorphism on the posets of open intervals, ordered by the Tchebyshev order. Under this isomorphism, the interval $[(-1,-2), (-(n+1), -(n+2))]$ goes into the interval $T_{n-1} = [(-1,1), (-n, -(n+1))]$. ◇

4 The geometric significance of the Tchebyshev poset T_n

In this section we show that the saturated chains of T_n provide a triangulation of the boundary of the n-dimensional cross-polytope. Let $e_1, \ldots, e_n$ be the basis vectors of an n-dimensional Euclidean space. The n-dimensional cross-polytope is the convex hull of the vectors $\pm e_1, \ldots, \pm e_n$. It has $2n$ vertices and 2^n facets, each facet being the convex hull of $\varepsilon_1 \cdot e_1, \ldots, \varepsilon_n \cdot e_n$ for some $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$.

We may use T_n to construct a triangulation of the boundary of the n-dimensional cross-polytope as follows. Label the vertex $\varepsilon_i \cdot e_i$ with $(\varepsilon_i \cdot i, -(n+1)) \in T_n$. Assuming $i < j$, label the midpoint of the edge connecting $\varepsilon_i \cdot e_i$ and $\varepsilon_j \cdot e_j$ with $(\varepsilon_i \cdot i, \varepsilon_j \cdot j)$. We define the convex hull of a set of labeled points to be a face in our triangulation if and only if the corresponding set of labels forms an increasing chain in the Tchebyshev poset T_n. In other words, we consider a geometric realization of the *order complex* of $T_n \setminus \{(-1,1), (-(n+1), -(n+2))\}$).

Theorem 4.1 *The geometric realization of the order complex of $T_n \setminus \{(-1,1), (-(n+1), -(n+2))\}$ described above provides a triangulation of the boundary of the n-dimensional cross-polytope.*

Figure 2 shows the triangulation of the octahedron induced by T_3. Invisible edges are indicated with a dotted line, except for the invisible edge connecting $(1,2)$ with $(3,-4)$, which is completely covered by visible edges. The fact that the order complex of $T_3 \setminus \{(-1,1), (-4,-5)\}$ is realized as a triangulation of the boundary of the octahedron is obvious from Figure 2. To prove Theorem 4.1 for an arbitrary n, observe first that every saturated chain of T_n is of the form

$$\widehat{0} = (-1,1) \prec (\varepsilon_1 1, \varepsilon_2 2) \prec (x_2, \varepsilon_3 3) \prec \cdots (x_{n-1}, \varepsilon_n n) \prec (x_n, -(n+1)) \prec (-(n+1), -(n+2)) = \widehat{1},$$

and no matter what the elements $x_2, \ldots, x_n$ are, if $|x_i| = j$ then the sign of x_i must be ε_j. Therefore the correspondingly labeled vertices all belong to the convex hull of $\varepsilon_1 e_1, \ldots, \varepsilon_n e_n$, that is, to the same facet of the boundary of the cross-polytope. Without loss of generality we may assume

$$\varepsilon_1 = \cdots = \varepsilon_n = 1, \tag{2}$$

and we only need to show that the geometric realization of all saturated chains satisfying this condition induce a triangulation of the convex hull of $e_1, \ldots, e_n$. Observe finally that the saturated chains satisfying condition (2) are exactly the saturated chains in the Tchebyshev poset of the chain $\widehat{0} = 1 < 2 < \cdots < n+1 < n+2 = \widehat{1}$. This motivates to introduce the following "unsigned version" of the Tchebyshev posets.

Definition 4.2 *Consider the set of nonnegative integers $\mathbb{N}$ ordered by the usual $<$ relation. We define the* unsigned Tchebyshev poset U_n *to be the interval* $[(0,1), (n+1, n+2)]$ *in the Tchebyshev poset* $T(\mathbb{N})$ *of* $\mathbb{N}$.

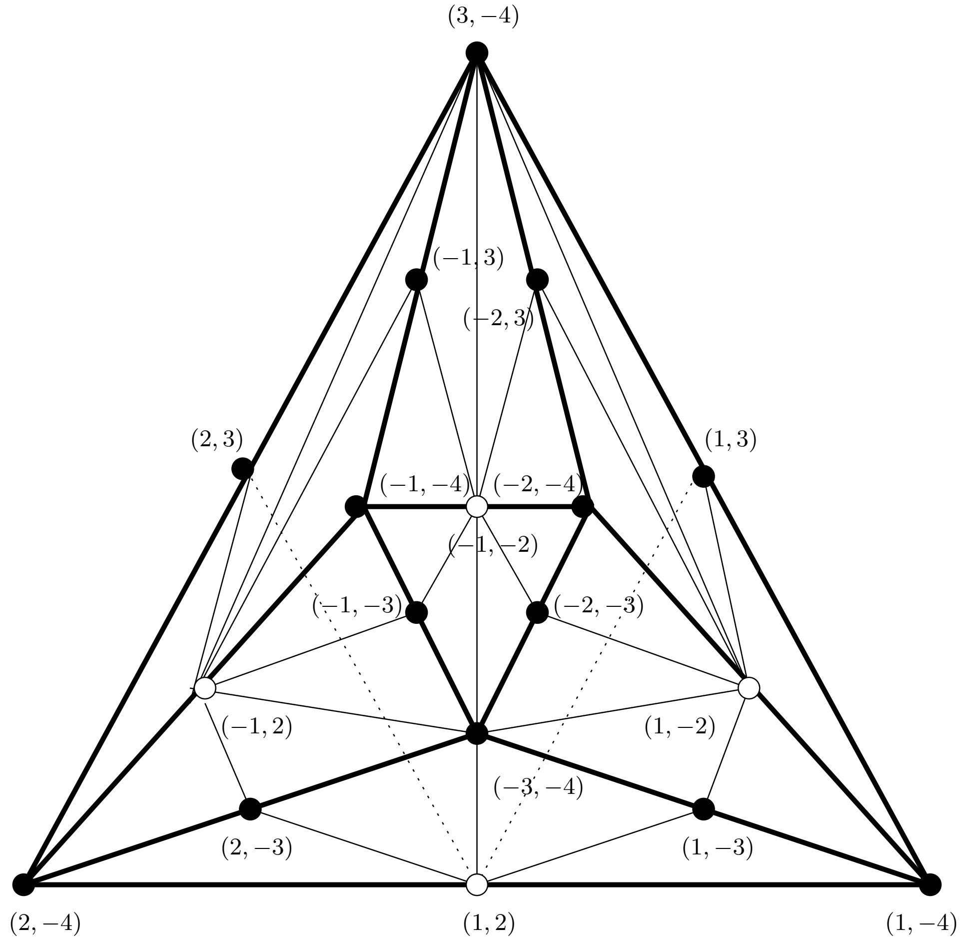

Figure 2: The triangulation of the octahedron induced by T_3

Figure 3 represents the unsigned Tchebyshev poset U_3. Note that we chose our indices in such a way that the rank of U_n becomes $n+1$, equal to the rank of T_n. The faces of the octahedron are 2-dimensional, and the unsigned Tchebyshev poset used in their triangulation will be U_2. We are including an illustration of U_3 here because from this picture a skilled reader will understand how to graph U_n in general. As noted above, the saturated chains satisfying (2) are the ones belonging to the Tchebyshev poset of $1 < \cdots < n+2$ which is isomorphic to the Tchebyshev poset of $0 < \cdots < n+1$, that is, to U_{n-1}. It is sufficient to show that the saturated chains of U_{n-1} induce a triangulation of the $(n-1)$-simplex, and thus we only need to show the following "unsigned version" of Theorem 4.1.

Proposition 4.3 *Consider an $(n+1)$-dimensional Euclidean space with basis $\{e_0, \ldots, e_n\}$, and the unsigned Tchebyshev poset U_n. Let $\triangle_n$ be the convex hull of $e_0, e_1, \ldots, e_n$. Label the vertex e_i of $\triangle_n$ with $(i, n{+}1)$, and label the midpoint of the edge connecting e_i and e_j with (i, j), whenever $0 \le i < j \le n$ holds. Then the convex hulls of points whose labels are the increasing chains in $U_n \setminus \{(n+1, n+2)\}$ provide a triangulation of the n-simplex $\triangle_n$.*

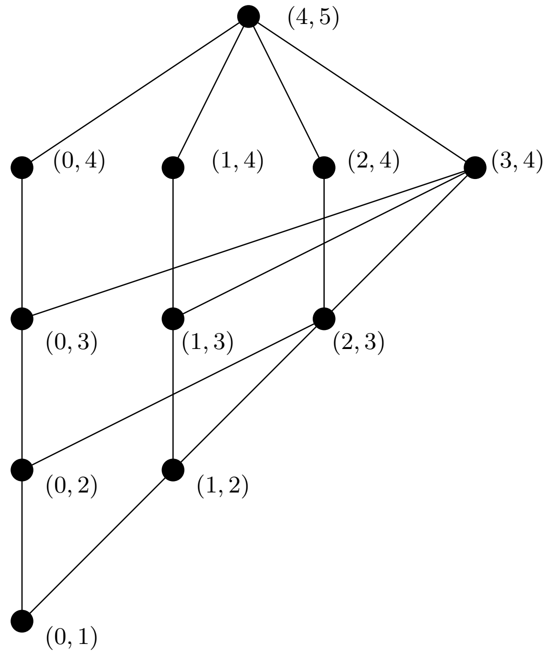

Figure 3: The unsigned Tchebyshev poset U_3

Proof: We proceed by induction on n. For $n = 1$ the simplex $\triangle_1$ is the line segment connecting e_0 and e_1, and U_1 consist of $(0,1) < (0,2),(1,2) < (2,3)$. The label of e_0 is $(0,2)$, the label of e_1 is $(1,2)$ and the label of the midpoint $(e_0+e_1)/2$ is $(0,1)$. The two saturated chains of $U_1 \setminus \{(2,3)\}$ correspond to the line segments from e_0 to $(e_0+e_1)/2$ and from $(e_0+e_1)/2$ to e_1. So in this case we obtain a (one-dimensional) "triangulation".

Assume that our proposition holds for some n and consider $\triangle_{n+1}$ and U_{n+1}. The minimum element $(0,1)$ is covered by two atoms: $(0,2)$ and $(1,2)$ in U_{n+1}. It is easy to see that, in analogy to Proposition 3.7, the intervals $[(0,2),(n+2,n+3)]$ and $[(1,2),(n+2,n+3)]$ are isomorphic to U_n. Let us also note that to obtain the first interval from U_n we must replace every positive i with $i+1$ (leaving 0 unchanged) while the second interval is obtained by increasing every i by 1. Hence, by our induction hypothesis, the increasing chains from $[(0,2),(n+2,n+3)] \setminus \{(n+2,n+3)\}$ induce a triangulation of the convex hull of $\{e_0,e_2,\ldots,e_{n+1}\}$ while the increasing chains from $[(1,2),(n+2,n+3)] \setminus \{(n+2,n+3)\}$ induce a triangulation of the convex hull of $\{e_1,e_2,\ldots,e_{n+1}\}$. By adding $(0,1)$ to any of these chains we obtain a triangulation of the convex hull of $\{(e_0+e_1)/2,e_0,e_2,\ldots,e_{n+1}\}$ and of $\{(e_0+e_1)/2,e_1,e_2,\ldots,e_{n+1}\}$. The union of these two simplices is the convex hull of $\{e_0,e_1,e_2,\ldots,e_{n+1}\}$. $\diamond$

An important corollary of Theorem 4.1 is the following.

Corollary 4.4 *The dual of the partially ordered set T_n is the partially ordered set of a CW complex* *that is homeomorphic to an n-ball.*

In fact, consider the triangulation of the boundary of the n-dimensional cross-polytope induced by $T_n \setminus \{(-1,1),(-(n+1),-(n+2))\}$. By associating the center of the cross-polytope to $(-1,1)$ we may easily convince ourselves that $T_n \setminus \{(-(n+1),-(n+2))\}$ induces a triangulation of the n-dimensional cross-polytope. Associate to each $(\varepsilon i, \eta j) \in T_n \setminus \{(-(n+1),-(n+2))\}$ the convex hull of all vertices represented by some $(\varepsilon' i', \eta' j') \geq (\varepsilon i, \eta j)$. Thus we associate the entire cross-polytope to $(-1,1)$, the union of some 2^{n-2} facets of its boundary to each $(\varepsilon 1, \eta 2)$, and so on, a vertex to each $(\varepsilon i, -(n+1))$. It is easy to verify that the convex hulls associated to the elements of $T_n \setminus \{(-(n+1),-(n+2))\}$ form the nonempty faces of a CW-complex, ordered by reverse inclusion. Since they arose as unions of faces from the triangulation of a polytope, the resulting CW-complex is homeomorphic to a ball.

Definition 4.5 *An n-dimensional* dual Tchebyshev cell *is a CW-complex whose partially ordered set of faces is isomorphic to the dual of T_n.*

Repeated use of Proposition 3.7 yields the following.

Corollary 4.6 *Every face of a dual Tchebyshev cell is a dual Tchebyshev cell.*

To conclude this section let us note that for a graded poset P, the order complex of $P \setminus \{\widehat{0}, \widehat{1}\}$ is *balanced*, that is the dimension of the complex plus one color suffices to color the vertices of the complex in such a way that no face contains two vertices of the same color. In fact, the rank function may be used to assign colors. Hence the Tchebyshev poset T_n induces a balanced triangulation of the n-dimensional cross-polytope, using less simplices than barycentric subdivision, the "usual way" to create a balanced triangulation. On Figure 2 we used white circles to denote those midpoints of edges whose color should be 2, while the remaining midpoints of edges should have color 3 and the vertices of the octahedron should have color 1. For more information on balanced simplicial complexes see Stanley's book [21] or his paper [20].

5 A shelling of the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$

The fact that the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ may be geometrically realized as a triangulation of the boundary of a convex polyhedron suggests that this order complex may be shellable. After all, the first shelling was originally introduced as an enumeration of the facets of convex polytope by Bruggeser and Mani in [11]. In this section we introduce an FA-labeling for T_n that has the additional property that for each saturated chain C the minimal elements of the associated family of intervals $\mathcal{I}_C$ of each saturated chain C is a collection of singletons. Hence, by Lemma 1.1 we obtain that the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ is shellable.

Every element of T_n below coatom-level is covered by at most 4 elements, and our labeling λ assigns to each edge a label from the totally ordered set $1 < 2 < 3 < 4$. Using the operators introduced in Definition 3.2 every element covering $(x, y) \in T_n$, may be written in exactly one of the following four forms: $R^-(x, y)$, $R^+(x, y)$, $L^-(x, y)$, and $L^+(x, y)$. Using this information we define our labeling as follows.

Definition 5.1 *(i) If the sign of y is positive, we set*

$$\lambda((x, y), R^-(x, y)) = 1,\ \lambda((x, y), R^+(x, y)) = 2,\ \lambda((x, y), L^-(x, y)) = 3,\ \lambda((x, y), L^+(x, y)) = 4.$$

(ii) If the sign of y as well as the sign of x is negative, we set

$$\lambda((x, y), L^-(x, y)) = 1,\ \lambda((x, y), R^-(x, y)) = 2,\ \lambda((x, y), L^+(x, y)) = 3,\ \lambda((x, y), R^+(x, y)) = 4.$$

(iii) If the sign of y is negative but the sign of x is positive, we set

$$\lambda((x, y), L^-(x, y)) = 1,\ \lambda((x, y), L^+(x, y)) = 2,\ \lambda((x, y), R^-(x, y)) = 3,\ \lambda((x, y), R^+(x, y)) = 4.$$

If the rank of (x, y) is $n - 1$ then only $L^-(x, y)$ and $R^-(x, y)$ exist, if (x, y) is a coatom then only $L^-(x, y)$ exists. Note that the labeling λ is an edge-labeling, that is, it does not depend on a choice of a maximal chain containing the edge.

Theorem 5.2 *The FA-labeling introduced in Definition 5.1 has the property that if we enumerate the saturated chains of T_n in any order extending the lexicographic order on their labels, the minimal intervals in the associated interval system $\mathcal{I}_C$ are singletons.*

The poset T_3 is represented on Figure 1 in such a way that for most (x, y) the labeling λ numbers the edges covering (x, y) from left to right, in increasing order. The only three exceptions are: $(1, 2)$, for which the order of labels is 3,4,1,2; $(2, 3)$ and $(1, 3)$, both of which are covered by two edges, labeled from left to right by 2, 1, in this order. It is very important to note that there is no "rising chain" in the interval $[(-1, -2), (3, -4)]$, hence our labeling is *not a CR-labeling.*

We prove Theorem 5.2 by establishing the validity of a sequence of lemmata. In each of the following lemmata we consider the labeling λ given in Definition 5.1, an arbitrary saturated chain $C : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_n, y_n) < \widehat{1}$, and we assume that $[i, j]$ is an arbitrary minimal interval of $\mathcal{I}_C$. We will be done with the proof of Theorem 5.2 once we have shown that $i = j$ must hold. By definition, the fact that $[i, j]$ belongs to $\mathcal{I}_C$ implies that there is an "earlier" saturated chain

$$C' : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_{i-1}, y_{i-1}) \prec (x'_i, y'_i) \prec \cdots \prec (x'_j, y'_j) \prec (x_{j+1}, y_{j+1}) \prec \cdots (x_n, y_n) \prec \widehat{1}$$

differing from C exactly at the ranks belonging to $[i,j]$.

Lemma 5.3 *Under our assumptions either $i=j$, or $(x_{j+1},y_{j+1}) = L^{\mathrm{sign}(y_{j+1})}(x_j,y_j)$ holds. In the latter case we must also have $(x_{k+1},y_{k+1}) = R^{\mathrm{sign}(y_{k+1})}(x_k,y_k)$ for $i+1 \le k < j$.*

Proof: Assume first that $(x_{j+1},y_{j+1}) = R^{\mathrm{sign}(y_{j+1})}(x_j,y_j)$ holds. Then, by Lemma 3.3, we must also have that the element (x_{j+1},y_{j+1}) covering (x'_j,y'_j) must be of the form $(x_{j+1},y_{j+1}) = R^{\mathrm{sign}(y_{j+1})}(x'_j,y'_j)$. Since the operators R^+ and R^- do not change the first coordinate, $x_j = x'_j = x_{j+1}$ follows. This also implies that the only way for (x_j,y_j) to be different from (x'_j,y'_j) is $y'_j = -y_j$. Now we may use Lemma 3.4 to conclude from $(x'_{j-1},y'_{j-1}) < (x'_j,y'_j)$ that $(x'_{j-1},y'_{j-1}) < (x_j,y_j)$. But then, unless $i=j$, the saturated chain

$$C_1 : \widehat{0} \prec (x_1,y_1) \prec \cdots \prec (x_{i-1},y_{i-1}) \prec (x'_i,y'_i) \prec \cdots \prec (x'_{j-1},y'_{j-1}) \prec (x_j,y_j) \prec \cdots (x_n,y_n) \prec \widehat{1}$$

which is "earlier" than C, demonstrates the fact that $[i,j-1]$ belongs to $\mathcal{I}_C$, in contradiction with the minimality of $[i,j]$. This contradiction is avoided only if $i=j$.

From now on we may assume $(x_{j+1},y_{j+1}) = L^{\mathrm{sign}(y_{j+1})}(x_j,y_j)$. Let k be the smallest integer satisfying $k \ge i+1$ and $(x_{k+1},y_{k+1}) = L^{\mathrm{sign}(y_{k+1})}(x_k,y_k)$. For this k we have

$$(x_{k+1},y_{k+1}) = L^{\mathrm{sign}(y_{k+1})} L^{\mathrm{sign}(x_{k+1})}(x'_{k-1},y'_{k-1})$$

by Lemma 3.5. Hence the saturated chain

$$\begin{aligned} C_2 : \widehat{0} &\prec (x_1,y_1) \prec \cdots \prec (x_{i-1},y_{i-1}) \prec (x'_i,y'_i) \prec \cdots \prec (x'_{k-1},y'_{k-1}) \prec L^{\mathrm{sign}(x_{k+1})}(x'_{k-1},y'_{k-1}) \prec \\ &\prec (x_{k+1},y_{k+1}) \prec \cdots \prec (x_n,y_n) \prec \widehat{1} \end{aligned}$$

which is "earlier" than C demonstrates $[i,k] \in \mathcal{I}_C$. By the minimality of $[i,j]$ we must have $k=j$, therefore every (x_{k+1},y_{k+1}) for $i+1 \le k < j$ must be in the image of some R-operator. ◇

Lemma 5.4 *Under our assumptions $j \le i+1$ holds.*

Proof: Assume, by way of contradiction, $j \ge i+2$. In this case, by Lemma 5.3, we have $(x_{j+1},y_{j+1}) = L^{\mathrm{sign}(y_{j+1})}(x_j,y_j)$ and $(x_{k+1},y_{k+1}) = R^{\mathrm{sign}(y_{k+1})}(x_k,y_k)$ for $i+1 \le k < j$. As a consequence of $(x_{j+1},y_{j+1}) = L^{\mathrm{sign}(y_{j+1})}(x_j,y_j)$ we must have

$$x_{j+1} = y_j \tag{3}$$

We claim that in this situation $y_{i+1}, y_{i+2}, \ldots, y_{j-1}$ must all have negative sign. In the contrary event, let m be the smallest integer satisfying $\mathrm{sign}(y_m) = +$ and $i+2 \le m \le j-1$. Consider the saturated chain

$$C_3 : \widehat{0} \prec (x_1,y_1) \prec \cdots \prec (x_{m-1},y_{m-1}) \prec (x_m,-y_m) \prec (x_{m+1},y_{m+1}) \prec \cdots \prec (x_n,y_n) \prec \widehat{1}.$$

This is a saturated chain by Lemma 3.4, and because of

$$(x_{m+1}, y_{m+1}) = R^{\mathrm{sign}(y_{m+1})}(x_m, y_m) = R^{\mathrm{sign}(y_{m+1})}(x_m, -y_m).$$

Since y_m is positive, either we have $(x_m, y_m) = R^+(x_{m-1}, y_{m-1})$ or $(x_m, y_m) = L^+(x_{m-1}, y_{m-1})$. In the first case $(x_m, -y_m) = R^-(x_{m-1}, y_{m-1})$, in the second $(x_m, y_m) = L^-(x_{m-1}, y_{m-1})$. Since in our labeling the label of $((x, y), R^-(x, y))$ is always less than the label of $((x, y), R^+(x, y))$ and the analogous statement holds also for the operators L^- and L^+, the chain C_3 defined above is "earlier" than C, and demonstrates $[m, m] \in \mathcal{I}_C$, in contradiction with the minimality of $[i, j]$. This contradiction proves that $y_{i+1}, y_{i+2}, \ldots, y_{j-1}$ must have negative sign. In particular, the sign of y_{j-1} is negative. Consider now the saturated chain

$$C_4 : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_{j-1}, y_{j-1}) \prec (y_{j-1}, y_j) \prec (x_{j+1}, y_{j+1}) \prec \cdots \prec (x_n, y_n) \prec \widehat{1}.$$

This is a saturated chain since $(y_{j-1}, y_j) = L^{\mathrm{sign}(y_j)}(x_{j-1}, y_{j-1})$ obviously holds, and we also have $(x_{j+1}, y_{j+1}) = L^{\mathrm{sign}(y_{j+1})}(y_{j-1}, y_j)$ by (3). Since y_{j-1} is negative, the label of

$$((x_{j-1}, y_{j-1}), (y_{j-1}, y_j)) = ((x_{j-1}, y_{j-1}), L^{\mathrm{sign}(y_j)}(x_{j-1}, y_{j-1}))$$

is less than the label of

$$((x_{j-1}, y_{j-1}), (x_j, y_j)) = ((x_{j-1}, y_{j-1}), R^{\mathrm{sign}(y_j)}(x_{j-1}, y_{j-1})).$$

Therefore C_4 is "earlier" than C, and $[j, j]$ belongs to $\mathcal{I}_C$, in contradiction with the minimality of $[i, j]$. This contradiction may only be avoided if $j \leq i + 1$. $\diamond$

The last part of the proof of Lemma 5.4 also allows to observe the following.

Lemma 5.5 *If $j > i$ then $j = i + 1$ and y_i has positive sign.*

Proof: The fact that $j = i + 1$ is a direct consequence of Lemma 5.4. Assume that y_i has negative sign. If $(x_{i+1}, y_{i+1}) = R^{\mathrm{sign}(y_{i+1})}(x_i, y_i)$ then we may reach a contradiction similar to the one in the last part of the proof of Lemma 5.4 by considering the saturated chain

$$C_5 : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_i, y_i) \prec (y_i, y_{i+1}) \prec (x_{i+2}, y_{i+2}) \prec \cdots \prec (x_n, y_n) \prec \widehat{1}$$

which is earlier than C and demonstrates $[i + 1, i + 1] \in \mathcal{I}_C$, a contradiction with the minimality of $[i, j] = [i, i + 1]$. Hence we may assume $(x_{i+1}, y_{i+1}) = L^{\mathrm{sign}(y_{i+1})}(x_i, y_i)$. Since y_i is negative, either $(x_i, y_i) = R^-(x_{i-1}, y_{i-1})$ or $(x_i, y_i) = L^-(x_{i-1}, y_{i-1})$ holds. Let (x_i^*, y_i^*) be that element of $\{R^-(x_{i-1}, y_{i-1}), L^-(x_{i-1}, y_{i-1})\}$, which is different from (x_i, y_i). Consider the saturated chain

$$C_6 : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_{i-1}, y_{i-1}) \prec (x_i^*, y_i^*) \prec (x_{i+1}, y_{i+1}) \prec \cdots \prec (x_n, y_n) \prec \widehat{1}$$

The fact that (x_i^*, y_i^*) is less than (x_{i+1}, y_{i+1}) follows from the fact that $L^{\text{sign}(y_{i+1})}(x_i^*, y_i^*))$ belongs to $\{L^{\text{sign}(y_{i+1})}R^-(x_{i-1}, y_{i-1}), L^{\text{sign}(y_{i+1})}L^-(x_{i-1}, y_{i-1})\}$, a singleton by Lemma 3.6, and so it is also equal to $(x_{i+1}, y_{i+1}) = L^{\text{sign}(y_{i+1})}(x_i, y_i))$ which belongs to the same singleton. No matter what the signs of x_{i-1} and y_{i-1} are, one of $((x_{i-1}, y_{i-1}), L^-(x_{i-1}, y_{i-1}))$ and $((x_{i-1}, y_{i-1}), R^-(x_{i-1}, y_{i-1}))$ has label 1. That cover relation can not be $((x_{i-1}, y_{i-1}), (x_i, y_i))$ since otherwise $[i, j] \in \mathcal{I}_C$ is not possible. Hence C_6 is an "earlier" saturated chain than C, demonstrating $[i, i] \in \mathcal{I}_C$. We have reached again a contradiction with the minimality of $[i, j]$. $\diamond$

Lemma 5.6 *Under our assumptions only $j = i$ is possible.*

Proof: Assume by way of contradiction that $j > i$ holds. By Lemma 5.5, we then must have $j = i+1$ and the sign of y_i is positive. Moreover, by Lemma 5.3, we must have

$$(x_{i+2}, y_{i+2}) = L^{\text{sign}(y_{i+2})}(x_{i+1}, y_{i+1}).$$

From here on we distinguish two cases, depending on whether (x_{i+1}, y_{i+1}) is in the image of some L-operator or of some R-operator.

Assume first that (x_{i+1}, y_{i+1}) is in the image of an L-operator, then by Lemma 3.3, we must have $(x_{i+1}, y_{i+1}) = L^{\text{sign}(y_{i+1})}(x_i, y_i)$. In this case consider the saturated chain

$$C_7 : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_i, y_i) \prec R^{\text{sign}(y_{i+1})}(x_i, y_i) \prec (x_{i+2}, y_{i+2}) \prec \cdots \prec (x_n, y_n) \prec \widehat{1}.$$

This is a saturated chain since $L^{\text{sign}(y_{i+2})}R^{\text{sign}(y_{i+1})}(x_i, y_i) = L^{\text{sign}(y_{i+2})}R^{\text{sign}(y_{i+1})} = (x_{i+2}, y_{i+2})$, and the label of $((x_i, y_i), R^{\text{sign}(y_{i+1})}(x_i, y_i))$ is smaller than the label of $((x_i, y_i), L^{\text{sign}(y_{i+1})}(x_i, y_i))$, since y_i is positive. Hence C_7 is an "earlier" chain, exhibiting the fact that $[i+1, i+1] \in \mathcal{I}_C$, in contradiction with the assumed minimality of $[i, j]$.

Thus from now on we may assume $(x_{i+1}, y_{i+1}) = R^{\text{sign}(y_{i+1})}(x_i, y_i)$. Since the sign of y_i is positive, either $(x_i, y_i) = L^+(x_{i-1}, y_{i-1})$ or $(x_i, y_i) = R^+(x_{i-1}, y_{i-1})$ holds. In the first case the saturated chain

$$C_8 : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_{i-1}, y_{i-1}) \prec L^-(x_{i-1}, y_{i-1}) \prec (x_{i+1}, y_{i+1}) \prec \cdots \prec (x_n, y_n) \prec \widehat{1},$$

in the second case the saturated chain

$$C_9 : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_{i-1}, y_{i-1}) \prec R^-(x_{i-1}, y_{i-1}) \prec (x_{i+1}, y_{i+1}) \prec \cdots \prec (x_n, y_n) \prec \widehat{1},$$

is "earlier" than C, therefore we reach the contradiction $[i, i] \in \mathcal{I}_C$ in either case. (The last part of Lemma 3.6 may be used to verify that C_8 and C_9 are saturated chains.) $\diamond$

6 "Descents" as a function of the LR-operators

In this we identify the "descent set" of each saturated chain C, that is the set of those $i \in [1, n]$ for which the singleton $[i, i]$ belongs to $\mathcal{I}_C$. We facilitate expressing our findings by observing that every saturated chain of T_n may be uniquely described as a sequence of words of length $n+1$ using letters from the alphabet $\{L^-, L^+, R^-, R^+\}$ subject to the restriction that the penultimate letter must be negative, and the last letter must be L^-. Here the i-th letter in the code of the chain will stand for the operator that takes (x_{i-1}, y_{i-1}) into (x_i, y_i). For example, the saturated chain

$$\widehat{0} = (-1, 1) \prec (1, -2) \prec (1, 3) \prec (1, -4) \prec (-4, -5) = \widehat{1}$$

in T_3 corresponds to the word $L^-R^+R^-L^-$. (If the reader wants to think in terms of composing operators then, unfortunately, at this point one would need to write the operators *after* the argument, which seems to be less elegant overall. However, this is the encoding that allows an easy transition to cd-words as they are usually defined in the literature of Eulerian posets.)

Definition 6.1 *We call the word associated to a saturated chain of T_n in the above described way the* LR-code *of the saturated chain.*

In particular, the number of saturated chains in T_n is $2 \cdot 4^{n-1}$. What may also take some getting-used-to for the first time reader, that the descents will not be associated to the letters of our words, but to the *gaps* between the letters and before the first letter. In fact the element (x_i, y_i) of rank i in a saturated chain $C : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_n, y_n) < \widehat{1}$ corresponds to the gap between letter (at position i) of the operator taking (x_{i-1}, y_{i-1}) into (x_i, y_i) and letter (at position $(i+1)$) of the operator taking (x_i, y_i) into (x_{i+1}, y_{i+1}). We may mark the descents using vertical bars between the letters. For the example above we will see that the only descent occurs at rank 2, thus we may write $L^-R^+|R^-L^-$ to mark that descent.

Theorem 6.2 *Consider the shelling described in section 5. Given a saturated chain $C : \widehat{0} \prec (x_1, y_1) \prec \cdots \prec (x_n, y_n) < \widehat{1}$ of T_n, the singleton $[i, i]$ belongs to $\mathcal{I}_C$ if and only if one of the following holds for its LR-code:*

(i) The letter at position $(i-1)$ is negative, the letter at position i is an R-operator, the letter at position $(i+1)$ is an L-operator (pattern $\ldots X^-R^\varepsilon|L^\eta \ldots$),

(ii) the letter at position $(i-1)$ is positive, the letter at position i is an L-operator, the letter at position $(i+1)$ is an L-operator (patterns $\ldots X^+L^\varepsilon|L^\eta \ldots$ and $L^\varepsilon|L^\eta \ldots$), or

(iii) the letter at position i is positive, and the letter at position $(i+1)$ is an R-operator (pattern $\ldots X^+|R^\varepsilon \ldots$).

Proof: The singleton $[i,i]$ belongs to $\mathcal{I}_C$ if and only if there exists an "earlier" saturated chain C' : $\widehat{0} \prec (x_1, y_1) \prec \cdots (x_{i-1}, y_{i-1}) \prec (x'_i, y'_i) \prec (x_{i+1}, y_{i+1}) \prec (x_n, y_n) < \widehat{1}$. Equivalently there must be an element (x'_i, y'_i) strictly between (x_{i-1}, y_{i-1}) and (x_{i+1}, y_{i+1}) such that the label of $((x_{i-1}, y_{i-1}), (x'_i, y'_i))$ is less than the label of $((x_{i-1}, y_{i-1}), (x_i, y_i))$.

Case 1: (x_{i+1}, y_{i+1}) is in the image of some L-operator.
In this case we must have $(x_{i+1}, y_{i+1}) = L^{\text{sign}(y_{i+1})}(x_i, y_i) = L^{\text{sign}(y_{i+1})}(x'_i, y'_i)$, and $y_i = y'_i = x_{i+1}$. Since $y_i = y'_i$, both (x_i, y_i) and (x'_i, y'_i) are in the image of some operator with the same sign $\text{sign}(y_i)$. Hence the set $\{(x_i, y_i), (x'_i, y'_i)\}$ is equal to the set $\{R^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}), L^{\text{sign}(y_i)}(x_{i-1}, y_{i-1})\}$. There are two possible ways to match the elements in these two sets. The first possibility is

$$(x_i, y_i) = R^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}) \quad \text{and} \quad (x'_i, y'_i) = L^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}).$$

In this case, the saturated chain C' is "earlier" if and only if the sign of y_{i-1} is negative. This is equivalent to saying that (x_{i-1}, y_{i-1}) is in the image of some operator X^- (where $X \in \{L, R\}$), and we recover the pattern $\ldots X^- R^\varepsilon | L^\eta \ldots$

The second possibility is

$$(x_i, y_i) = L^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}) \quad \text{and} \quad (x'_i, y'_i) = R^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}).$$

In this case, the saturated chain C' is "earlier" if and only if the sign of y_{i-1} is positive. This is equivalent to saying that (x_{i-1}, y_{i-1}) is in the image of some operator X^+ (where $X \in \{L, R\}$), or that $(x_{i-1}, y_{i-1} = (-1, +1)$ and we recover the patterns $\ldots X^+ L^\varepsilon | L^\eta \ldots$ and $L^\varepsilon | L^\eta \ldots$

Case 2: (x_{i+1}, y_{i+1}) is in the image of some R-operator.
In this case we must have $(x_{i+1}, y_{i+1}) = R^{\text{sign}(y_{i+1})}(x_i, y_i) = R^{\text{sign}(y_{i+1})}(x'_i, y'_i)$, and $x_i = x'_i = x_{i+1}$. The only way for (x_i, y_i) and (x'_i, y'_i) to be different is by y_i and y'_i having opposite signs. Hence the set $\{(x_i, y_i), (x'_i, y'_i)\}$ is either equal to $\{R^-(x_{i-1}, y_{i-1}), R^+(x_{i-1}, y_{i-1})\}$ or it is equal to $\{L^-(x_{i-1}, y_{i-1}), L^+(x_{i-1}, y_{i-1})\}$. Either way, C' is "earlier" than C exactly when (x_i, y_i) is in the image of the positive-signed operator. We recover the pattern $\ldots X^+ | R^\varepsilon \ldots$. ◇

Using Theorem 6.2 it is relatively easy to recover the flag h-vector of T_n, but for the sake of calculating the *cd*-index it is worth observing the following. The two patterns in case (ii) may be merged into one pattern if we note that $\widehat{0} = (-1, 1)$ has a positive second coordinate, and could be thought of as the "image of an operator L^+". So we may convert our codes for the saturated chains of T_n to words of length $n+1$ by adding an initial letter L^+ in front of each word. This will yield the *positive augmented LR-code* of the saturated chain. For example, for the saturated chain

$$\widehat{0} = (-1, 1) \prec (1, -2) \prec (1, 3) \prec (1, -4) \prec (-4, -5) = \widehat{1}$$

in T_3 we obtain the code $L^+ L^- R^+ R^- L^-$. Descents now correspond to the gaps between letters, except for the gap between the first two letters. The description of the descent-yielding patterns may be simplified to:

$$\dots X^{-}R^{\varepsilon}|L^{\eta}\dots, \quad \dots X^{+}L^{\varepsilon}|L^{\eta}\dots, \quad \text{or} \quad \dots X^{+}|R^{\varepsilon}\dots. \tag{4}$$

Let us observe next that, by Proposition 3.7, the poset T_n is also isomorphic to the interval $[(1,-2),\widehat{1}]$ of T_{n+1}, and so we may also enumerate the descents of the saturated chains of T_n by enumerating the descents of all saturated chains in the rooted interval $[(1,-2),\widehat{1}]_{\widehat{0}\prec(1,-2)}$ of T_{n+1}. It is easy to verify that this enumeration corresponds to the following encoding: we append an L^- in front of the LR-code of each saturated chain of T_n, and use the same list of patterns (4) to describe the position of the descents. We call the resulting long code the *negative augmented LR-code* of the saturated chain. For example, for the saturated chain

$$\widehat{0} = (-1,1) \prec (1,-2) \prec (1,3) \prec (1,-4) \prec (-4,-5) = \widehat{1}$$

in T_3 we obtain the code $L^-L^-R^+|R^-L^-$. Note that the descent set associated to this chain when using the negative augmented LR-code is different: it is $\{2\}$, as marked in the code. But if we sum up the descents associated to the negative augmented LR-code for all saturated chains of T_n, we obtain the same flag h-vector statistic, since we end up computing the flag h vector of the same partially ordered set. We may also use both codes and take the average.

Corollary 6.3 *The flag h-vector of T_n may be calculated as follows.*

1. *Write down all words $u = u_0u_1\cdots u_{n+1}$ of length $n+2$ using the alphabet $\{L^-, L^+, R^-, R^+\}$ satisfying $u_0 \in \{L^-, L^+\}$, $u_n \in \{L^-, R^-\}$, and $u_{n+1} = L^-$.*
2. *Number the gaps between the letters left to right with the numbers $1, \dots n$, omitting the first gap. (Gap number i follows the letter u_i.)*
3. *Mark the descents wherever any of the patterns listed in (4) occurs.*
4. *$h_S(T_n)$ is the half of the number of words for which exactly the gaps associated to the set $S \subseteq [1,n]$ are marked as descents.*

7 The cd-index of T_n

The use of Corollary 6.3 makes the calculation of the cd-index of T_n really easy.

Theorem 7.1 *The coefficient of the cd-word $w = c^{k_1}dc^{k_2}d\cdots dc^{k_{r+1}}$ in the cd-index of T_n is*

$$2^{\deg_d(w)}(k_1+1)\cdots(k_r+1).$$

Here $\deg_d(w)$ denotes the number of d's occurring in the cd-word w.

Proof: Consider the set $W_n^{\pm}$ of all words described in Corollary 6.3. The ab-index of T_n, using its definition and Corollary 6.3, may be described as follows. For each word $u_0 \cdots u_{n+1} \in W_n^{\pm}$ write a b below the gaps marked as descents, and a letter a to all other gaps, except for the gap between u_0 and u_1. One half times the sum of the associated ab-words over all words from $W_n^{\pm}$ yields the ab-index.

Switching the sign of any letter except for the sign of u_n and u_{n+1} takes $W_n^{\pm}$ into itself. Allowing all such sign switches induces a $\mathbb{Z}_2^n$-action on $W_n^{\pm}$. Each equivalence class under this action may be characterized by word of length $n+2$ on the (unsigned) alphabet $\{L, R\}$ starting and ending with the letter L. (The sign of the last two letters may be dropped, since they have to be negative, the sign of the other letters may be ignored due to the $\mathbb{Z}_2^n$-action.) We claim that each equivalence class contributes the constant multiple of exactly one cd-monomial w to the cd-index. The coefficient of the contributed cd-monomial will be $2^{\deg_d(w)-1}$. (Different equivalence classes may contribute the same $2^{\deg_d(w)}w$.) To see this, consider an arbitrary (signed) LR-word $u = u_0 \ldots u_{n+1} \in W_n^{\pm}$. Let us determine the effect of choosing the sign of u_i on the descent set where $0 \leq i \leq n$. Let us observe that the choice of the sign of u_n will not change the position of the descent, hence we may focus on the remaining u_i's, each of which is followed by at least two letters.

Case 1: u_{i+1} is of the form R^ε and u_{i+2} is of the form L^η. According to the list of descent patterns (4), depending the on the choice of the sign of u_i we see either the descent pattern $X^- R^\varepsilon | L^\eta$ or the descent pattern $X^+ | R^\varepsilon L^\eta$ at positions $i, i+1, i+2$, and the sign of u_i alone will determine which of the gaps numbered i and $i+1$ respectively is a descent. Note also that the case $i = 0$ is special, since the gap between u_0 and u_1 is ignored for the purposes of the descent statistics. Choosing the sign of u_i will have no effect whatsoever on the existence of a descent at any other gap. Hence summing over the sign choices of u_i will contribute a factor of $ab + ba = d$ covering the gaps numbered i and $i+1$ if $i > 0$ and a factor of $a + b = c$ for $i = 0$.

Case 2: both $u_{i+1} = L^\varepsilon$ and $u_{i+2} = L^\eta$ are signed L-letters. For these the sign of u_i will determine whether the descent pattern $X^+ L^\varepsilon | L^\eta$ occurs or not at positions $i, i+1, i+2$, and will have no effect on the existence of descents at any other positions. Hence summing over the sign choices of such a u_i will contribute a factor $a + b = c$ covering gap number $i + 1$.

Case 3: u_i is followed by to signed R-letters, $u_{i+1} = R^\varepsilon$ and $u_{i+2} = R^\eta$. The sign of such u_i's will determine alone whether there is a descent between u_i and u_{i+1} (according to the pattern $X^+ | R^\varepsilon$) and will have no effect whatsoever on the existence of any other descents. Hence such u_i's contribute a factor of $a + b = c$ covering gap number i. Again the case $i = 0$ is special, since the gap between u_0 and u_1 is ignored in the descent statistics.

Case 4: u_i is followed by $u_{i+1} = L^\varepsilon$ and $u_{i+2} = R^\eta$. In this case the sign of u_i has no bearing whatsoever on the existence of any descent, summing over the sign choices of u_i yields a factor of 2.

Observe also that that in any of the patterns listed in (4), switching the sign of the letter denoted by X makes the descent in question disappear. In fact, there is no descent between R^ε and L^η in a pattern $\ldots X^+R^\varepsilon L^\eta\ldots$, no descent between L^ε and L^η in the pattern $\ldots X^-L^\varepsilon L^\eta\ldots$ and no descent before R^ε in a pattern X^-R^ε. Moreover, in all patterns the letter X is followed by 2 letters, except for the last pattern, where we know that X^+ can not be u_n, since the last letter must always be L^-. This means that the existence of any descent is controlled by the sign of some u_i followed by at least two letters, and the four cases above cover the generation (and "disappearance") of all descents.

What we obtained is that the equivalence class of u contributes a multiple of that cd-word w in which the letters d cover exactly the gaps before and after those $u_{i+1}=R^\varepsilon$ satisfying $1\le i\le n-1$, which are followed by a $u_{i+2}=L^\eta$. The coefficient of the cd-word will be one half times 2 raised to the power of the number of those u_i's whose sign has no influence on the descent structure. Observe that the sign of every u_i associated to a letter d determines the existence of descents at two positions, making the role of of one u_i in determining the descent structure unnecessary, while the u_i's associated to a letter c determine the existence of a descent at one position. Hence the number of u_i's whose sign has no influence on the descent structure is the number of d's, and the coefficient of w is exactly $2^{\deg_d(w)-1}$.

Finally we need to answer the question, how many equivalence classes yield the same cd-word. Each equivalence class may be uniquely represented by a word $v=v_0\ldots v_{n+1}$ on the (unsigned) alphabet $\{L,R\}$ that satisfies $v_0=v_{n+1}=L$. From what was said above, it is clear that an equivalence class contributes $2^{\deg_d(w)}\cdot w$ for some cd word $w=c^{k_1}dc^{k_2}d\cdots dc^{k_{r+1}}$ if and only if the gaps before and after the letters v_i satisfying $v_i=R$, $i>2$, and $v_{i+1}=L$, mark exactly the beginnings of the d's in w. For example, the term $2^{2-1}\cdot dccdc$ is contributed by the following equivalence classes of saturated chains in T_7: $LLRLLLRLL$, $LLRLLRRLL$, $LLRLRRRLL$, $LRRLLLRLL$, $LRRLLRRLL$, and $LRRLRRRLL$. Here we may observe that whenever there is a c^{k_j} between two d's that corresponds to k_j letters between two RL patterns. These k_j letters may contain any number of R's between 0 and k_j, but once we fix this number there is a unique way of filling in the letters since we are not allowed to create a new RL pattern, and so the R's must be right-adjusted. This gives us (k_j+1) independent options. Concerning the letters before the first RL pattern marking a d we may make almost the same observation, but we also have to note that we are always allowed to change to choose the letter v_1 to be either R or L, since $v_1=R$, $v_2=L$ is an RL pattern that does not induce a letter d. For example, the term ccd is contributed by the following equivalence classes of saturated chains in T_4: $LLLLRL$, $LLLRRL$, $LLRRRL$, $LRRRRL$; $LRLLRL$, $LRLRRL$. If there is a c^{k_1} before the first d, then the corresponding RL pattern is preceded by $v_0v_1\cdots v_{k+2}$. Here $v_0=L$, we have 2 choices to setv_1, and k_1+1 choices to choose the number of (right-adjusted) R letters among $v_2\cdots v_{k+2}$. This yields a factor of $2\cdot(k_1+1)$. Note finally that all letters after the last RL pattern must be L's.

Therefore there are $2(k_1+1)\cdots(k_r+1)$ equivalence classes contributing to the same cd monomial $w=c^{k_1}dc^{k_2}d\cdots dc^{k_{r+1}}$, and each such class contributes $2^{\deg_d(w)-1}$ to its coefficient. $\diamond$

The cd-index of T_n for the first few values of n is provided in Table 7.

$$\begin{array}{rcl}
\Psi_{cd}(T_0) & = & 1 \\
\Psi_{cd}(T_1) & = & c \\
\Psi_{cd}(T_2) & = & c^2 + 2d \\
\Psi_{cd}(T_3) & = & c^3 + 4cd + 2dc \\
\Psi_{cd}(T_4) & = & c^4 + 6c^2d + 4cdc + 2dc^2 + 4d^2 \\
\Psi_{cd}(T_5) & = & c^5 + 8c^3d + 6c^2dc + 4cdc^2 + 2dc^3 + 8cd^2 + 8dcd + 4d^2c
\end{array}$$

Table 1: The cd-index of T_n for $n \leq 5$.

Remark 7.2 With some effort, it is also possible to deduce the cd-index formulas of this section from the ce-index formulas that will be shown in section 8, without using shelling. However, it is always an intriguing challenge to describe the cd-index of a class of "symmetric" posets using some sort of statistics on words. The earliest example of such a calculation may be found in Purtill's paper [19] who calculated the cd-index of the Boolean algebras using an ascent-descent statistics studied by Foata, Schutzenberger, and Strehl in [14], [15], [16], and [17]. The existence of such interactions still needs a better explanation.

8 The ce-index and the flag f-vector of T_n

As a consequence of Proposition 3.7, we may write and easy recursion formula for the ce-index of T_n.

Proposition 8.1 *The ce-index $\Psi_{ce}(T_n)$ of T_n satisfies the recursion formula*

$$\Psi_{ce}(T_n) = 2c\Psi_{ce}(T_{n-1}) - e^2\Psi_{ce}(T_{n-2}) \quad \textit{for } n \geq 2.$$

Proof: As it is known (see e.g. Stanley's paper [22, Formula (5)]), the ab-index of an Eulerian poset of rank $n+1$ may also be written as

$$\sum_{S \subseteq [1,n]} f_S \cdot v_S$$

where $v_S = v_1 \cdots v_n$ and

$$v_i = \begin{cases} b & \text{if } i \in S, \\ a - b & \text{if } i \notin S. \end{cases}$$

Let us apply this formula to T_n, and split the sum into two parts depending on whether $S \subseteq [1,n]$ contains 1 as an element.

Case 1: $1 \in S$. Since T_n has four atoms, and above each atom, according to Proposition 3.7, there is a copy of T_{n-1}, we obtain

$$\sum_{\substack{S\subseteq[1,n]\\1\in S}} f_S(T_n)\cdot v_S = 4b\Psi_{ab}(T_{n-1}).$$

Case 2: $1 \notin S$. All elements of a partial chain not containing any element of rank 1 are either above $(-1,-2)$ or $(+1,+2)$. If a partial chain is above both of these elements, then it contains no element of rank 2 and it is a partial chain in the Tchebyshev poset of

$$Q : -3, 3 < -4, 4 < \cdots < -n, n < -(n+1) < -(n+2)$$

that avoids the top element $(-(n+1), -(n+2))$. The Tchebyshev poset of $-3 < 3 < -4, 4 < \cdots < -n, n < -(n+1) < -(n+2)$ differs from $T(Q)$ only in having an extra minimum element $(-3,3)$ and this latter poset is isomorphic to T_{n-2}. Hence summing over all partial chains of $T(Q)$ avoiding its top element has the same effect as summing over all partial chains of T_{n-2} that avoid its top at bottom element. Therefore

$$\sum_{\substack{S\subseteq[1,n]\\1\notin S}} f_S(T_n)\cdot v_S = 2(a-b)\Psi_{ab}(T_{n-1}) - (a-b)^2\Psi_{ab}(T_{n-2}).$$

Adding up the contributions of all partial chains from both cases yields

$$\Psi_{ab}(T_n) = (4b+2a-2b)\Psi_{ab}(T_{n-1}) - (a-b)^2\Psi_{ab}(T_{n-2}) = 2(a+b)\Psi_{ab}(T_{n-1}) - (a-b)^2\Psi_{ab}(T_{n-2}),$$

and so we are done by $c = a+b$ and $e = a-b$. ◇

Corollary 8.2 *The substitution $c \mapsto x$, $e \mapsto 1$ sends the ce-index of T_n into the Tchebyshev polynomial $T_n(x)$.*

In fact, under this substitution, the $\Psi_{ce}(T_0)$ goes into 1, the $\Psi_{ce}(T_1)$ goes into x, and the recursion formula of Proposition 8.1 goes into the well-known recursion

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

of Tchebyshev polynomials. This corollary is by the way the one that motivated the name "Tchebyshev poset".

Remark 8.3 The substitution $c \mapsto x$, $e \mapsto 1$ sends d into $(x^2-1)/2$, and thus Theorem 7.1 yields the following formula for Tchebyshev polynomials:

$$T_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2m}\left(\frac{x^2-1}{2}\right)^m \cdot 2^m \sum_{\substack{S\subseteq[1,n]\\|S|=2m\\S\text{ even}}} \prod_{\substack{I\in\mathcal{I}([1,n]\setminus S)\\n\notin I}} (|I|+1).$$

Here $\mathcal{I}([1,n]\setminus S)$ stands for the collection of longest possible intervals whose disjoint union is $[1,n]\setminus S$. It is also well known that

$$T_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2m} \left(\frac{x^2-1}{2}\right)^m \cdot 2^m \binom{n}{2m}.$$

Comparing coefficients yields:

$$\binom{n}{2m} = \sum_{\substack{S\subseteq[1,n] \\ |S|=2m \\ S \text{ even}}} \prod_{\substack{I\in\mathcal{I}([1,n]\setminus S) \\ n\notin I}} (|I|+1).$$

Using Proposition 8.1 it is easy to obtain the following closed formula for the ce-index of T_n.

Theorem 8.4 *The coefficient of a ce-word u in $\Psi_{ce}(T_n)$ is $2^{\deg_c(u)-1}(-1)^{\deg_e(u)/2}$ if the last e in u is followed by at least one c, and $2^{\deg_c(u)}(-1)^{\deg_e(u)/2}$ otherwise. Equivalently*

$$L_S^{n+1}(T_n) = \begin{cases} 2^{n-|S|-1}(-1)^{|S|/2} & \text{if } S \text{ is even and } n \notin S, \\ 2^{n-|S|}(-1)^{|S|/2} & \text{if } S \text{ is even and } n \in S, \\ 0 & \text{if } S \text{ is not even.} \end{cases}$$

The proof is a straightforward induction on n and left to the reader.

We conclude this section with calculating the flag f-vector of T_n.

Proposition 8.5 *Assume that the elements of $S \subseteq [1,n]$ are $1 \le s_1 < s_2 < \cdots < s_k \le n$. Then*

$$f_S(T_n) = \begin{cases} 4^{|S|} s_1(s_2-s_1)(s_3-s_2)\cdots(s_k-s_{k-1}) & \text{if } n \notin S, \\ \frac{4^{|S|}}{2} s_1(s_2-s_1)(s_3-s_2)\cdots(s_k-s_{k-1}) & \text{if } n \in S. \end{cases}$$

Proof: The number $f_S(T_n)$ is the number of increasing chains of the form

$$(\varepsilon_1 j_1, \eta_1(s_1+1)) < (\varepsilon_2 j_2, \eta_2(s_2+1)) < \cdots < (\varepsilon_k j_k, \eta_k(s_k+1))$$

where the ε_i's and η_j's are signs. Assume first that $n \notin S$, i.e., $s_k < n$, and let us count the number of such increasing chains by filling it up from the bottom up. The possible values of j_1 are $1, 2, \ldots, s_1$. There are 4 ways to choose the signs ε_1 and η_1, so the first element of our chain may be chosen $4s_1$ ways. Suppose now we have already chosen $(\varepsilon_1 j_1, \eta_1(s_1+1))$, $(\varepsilon_2 j_2, \eta_2(s_2+1))$, $\ldots$, $(\varepsilon_{i-1} j_{i-1}, \eta_{i-1}(s_{i-1}+1))$, and let us count the number of ways to choose $(\varepsilon_i j_i, \eta_i(s_i+1))$. The possible values of j_i are: $j_{i-1}, s_{i-1}+1, s_{i-1}+2, \ldots, s_i$. If $j_i = j_{i-1}$, then we must have $\varepsilon_i = \varepsilon_{i-1}$ and if $j_i = s_{i-1}$ then we must have $\varepsilon_i = \eta_{i-1}$. In all other cases there are two ways to choose ε_i. There are always two ways to choose η_i, and so choosing $(\varepsilon_i j_i, \eta_i(s_i+1))$ contributes a factor of

$2+2+4(s_i - s_{i-1} - 1) = 4(s_i - s_{i-1} - 1)$. Multiplying the numbers of choices we made we obtain exactly the stated formula.

The only difference arising when $n \in S$ is that the sign η_k of $s_k + 1 = n + 1$ must be $-$, and so we have only half as many options to choose the last element of our increasing chain. $\diamond$

9 Extremal property of the Tchebyshev posets

The main result of this section is the following extremal property of the Tchebyshev poset T_d.

Theorem 9.1 *For $d \geq 1$, the f-vector $(f_{-1}, f_0, \ldots, f_{d-1})$ of an arbitrary $(d-1)$-dimensional simplicial complex satisfies the inequality*

$$\max_{-1 \leq x \leq 1} \left| \frac{2^d \sum_{j=0}^{d} f_{j-1} \left(\frac{x-1}{2}\right)^j}{f_{d-1}} \right| \geq \frac{1}{2^{d-1}}.$$

Equality holds for the order complex of $T_d \setminus \{\widehat{0}, \widehat{1}\}$.

As a first step towards the proof of Theorem 9.1, let us collect the terms of the polynomial $\sum_{j=0}^{d} f_{j-1} \left(\frac{x-1}{2}\right)^j$ and denote the coefficient of x^{d-i} by L_i. We call the vector $(L_0, \ldots, L_d)$ obtained this way the *L-vector* of the simplicial complex.

Definition 9.2 *The L-*vector *$(L_0, L_1, \ldots, L_d)$ of a $(d-1)$-dimensional simplicial complex is given by*

$$L_i = (-1)^{d-i} \sum_{j=d-i}^{d} \left(-\frac{1}{2}\right)^j f_{j-1} \binom{j}{d-i}.$$

Introducing $y = \frac{x-1}{2}$ allows to transcribe $\sum_{i=0}^{d} L_i x^{d-i} = \sum_{j=0}^{d} f_{j-1} \left(\frac{x-1}{2}\right)^j$ into

$$\sum_{i=0}^{d} L_i (2y+1)^{d-i} = \sum_{j=0}^{d} f_{j-1} y^j.$$

Comparing the coefficients of y^j yields

$$f_{j-1} = 2^j \sum_{i=0}^{d-j} \binom{d-i}{j} L_i.$$

In other words, the L-vector is a linearly equivalent encoding of the f-vector. It is easy to verify that for the order complex of an arbitrary graded partially ordered set satisfies

$$L_i = \sum_{|S|=i} L_S. \tag{5}$$

This equation may be derived from $f_{j-1} = \sum_{|T|=j} f_T$, using straightforward substitution into the definitions. As a consequence of equation (5) and Corollary 8.2 we obtain that for the order complex of T_d the polynomial $\sum_{j=0}^{d} f_{j-1} \left(\frac{x-1}{2}\right)^j$ is the Tchebyshev polynomial $T_d(x)$, with leading coefficient 2^{d-1}. For a general simplicial complex of dimension $(d-1)$ the leading coefficient of $\sum_{j=0}^{d} f_{j-1} \left(\frac{x-1}{2}\right)^j$ is $L_0 = \left(\frac{1}{2}\right)^d f_{d-1}$, and so the leading coefficient of

$$\frac{2^{2d-1} \sum_{j=0}^{d} f_{j-1} \left(\frac{x-1}{2}\right)^j}{f_{d-1}}$$

is 2^{d-1}. As it is well known, for polynomials $f(x)$ of degree d with leading coefficient 2^{d-1}, we have

$$\max_{-1 \leq x \leq 1} |f(x)| \geq 1$$

with equality when $f(x) = T_d(x)$. Therefore we have

$$\max_{-1 \leq x \leq 1} \left| \frac{2^{2d-1} \sum_{j=0}^{d} f_{j-1} \left(\frac{x-1}{2}\right)^j}{f_{d-1}} \right| \geq 1.$$

Dividing both sides by 2^{d-1} yields the statement of Theorem 9.1.

10 Concluding remarks

Perhaps the most important future use of Tchebyshev posets could be the investigation of a new special instance of Stanley's conjecture on the non-negativity of the *cd*-index of Gorenstein* partially ordered sets. In [24, Conjecture 2.1] Stanley made the conjecture that every Eulerian poset whose order complex has the Cohen-Macaulay property, i.e., every Gorenstein* poset, has a non-negative *cd*-index. In the same paper Stanley proved his conjecture (using spherical shelling) for boundary complexes of polytopes, and (by reducing the statement to known results about simplicial spheres) for simplicial partially ordered sets. In [12] R. Ehrenborg and myself gave an analogous description of the *cd*-index of Eulerian cubical posets. Using our description, the verification of Stanley's conjecture in the case of cubical posets is reduced to the question of non-negativity of Adin's h-vector (defined in [1]) for Cohen-Macaulay cubical complexes.

In analogy to the case of simplicial and cubical posets, one could restrict attention to those Eulerian partially ordered sets P for which every interval $[\widehat{0}, x]$ is isomorphic to the dual of a Tchebyshev poset T_n. If P is the face poset of a CW-complex then this complex is given by the property that each of its cells is a dual Tchebyshev cell.

Conjecture 10.1 *In the case of CW-complexes of dual Tchebyshev cells, Stanley's conjecture may be reduced to a non-negativity statement on an (appropriately defined) h-vector. Just like the simplicial h-vector, and Adin's cubical h-vector, this new h-vector will be a linear recoding of the f-vector of the complex.*

Two key issues are likely to make the treatment of such complexes more difficult than that of its simplicial or cubical "cousins". First, both simplicial and cubical h-vectors may be defined in relation to shellings. This is unlikely to happen to "complexes of dual Tchebyshev cells", since dual Tchebyshev cells themselves are non-polytopal, have only four facets in any dimension, thus the number of types of shelling components is not likely to grow with the dimension, while the dimension of the vector space generated by the f-vectors probably will. Second, the *Dehn-Sommerville equations* for cubical and simplicial spheres were already known to Grünbaum [18]. In terms of the cubical and simplicial h vectors, these equations may be rewritten as $h_i = h_{d-i}$. It is not yet known whether the dimension of the vector space of f-vectors of d-dimensional "spheres of dual Tchebyshev cells" would be essentially $d/2$. It is easy to write up that many (linearly independent) relations for the f-vector, but constructing that many examples with linearly independent f-vectors seems to be a daunting task, again because of the "lack of polytopality".

On the other hand "spheres of dual Tchebyshev cells" are still much more special than Gorenstein* posets in general. Proving Stanley's conjecture for them, together with the simplicial case could perhaps inspire a long-awaited "common generalization".

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