

# The Existence of Subspace Wavelet Sets

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## ABSTRACT

Let  $\mathcal{H}$  be a reducing subspace of  $L^2(\mathbb{R}^d)$ , that is, a closed subspace of  $L^2(\mathbb{R}^d)$  with the property that  $f(A^m t - \ell) \in \mathcal{H}$  for any  $f \in \mathcal{H}$ ,  $m \in \mathbb{Z}$  and  $\ell \in \mathbb{Z}^d$ , where  $A$  is a  $d \times d$  expansive matrix. It is known that  $\mathcal{H}$  is a reducing subspace if and only if there exists a measurable subset  $M$  of  $\mathbb{R}^d$  such that  $A^t M = M$  and  $\mathcal{F}(\mathcal{H}) = L^2(\mathbb{R}^d) \cdot \chi_M$ . Under some given conditions of  $M$ , it is known that there exist  $A$ -dilation subspace wavelet sets with respect to  $\mathcal{H}$ . In this paper, we prove that this holds in general.

*Keywords:* Frame, Wavelet, Frame Wavelet, Frame Wavelet Set, Fourier Transform.

## §1. Introduction

Let  $A$  be a  $d \times d$  real expansive matrix, i.e., a matrix with real entries whose eigenvalues are all of modules greater than one. Let  $L^2(\mathbb{R}^d)$  be the set of all square Lebesgue integrable functions in  $\mathbb{R}^d$  and let  $\mu$  be the Lebesgue

measure in  $\mathbb{R}^d$ . An *A-dilation wavelet* is a *single* function  $\psi \in L^2(\mathbb{R}^d)$  such that the set

$$\{|\det A|^{\frac{n}{2}}\psi(A^n t - \ell) : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$$

forms an orthonormal basis for  $L^2(\mathbb{R}^d)$ . The Fourier transform  $\mathcal{F}$  is defined by

$$(\mathcal{F}f)(s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(s \circ t)} f(t) d\mu, \quad (1)$$

for  $L^1$ -functions and can be extended to a unitary operator on  $L^2(\mathbb{R}^d)$ , where  $s \circ t$  denotes the real inner product. One important issue in the wavelet theory is how to construct various *A-dilation wavelets*, both for theoretical studies and for practical applications. It is proved in [5] that single function *A-dilation wavelets* exist for *every* expansive matrix *A*. The well-known and most widely applied *A-dilation wavelets* are the so called *MRA wavelets*, which are associated with and constructed from subspaces of  $L^2(\mathbb{R}^d)$  with certain special properties [3]. Unfortunately, these are not easy to come by in the higher dimension case. On the theoretical study side, a special kind of wavelets receives much of the attention from mathematicians due to the simplicity of their forms. These are the *A-dilation wavelets* of the form  $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$ , where *E* is a measurable set in  $\mathbb{R}^d$ . We call such a set *E* an *A-dilation wavelet set*. It is proved in [5,6] that for any  $d \times d$  expansive matrix *A*, there exist *A-dilation wavelet sets* in  $\mathbb{R}^d$ . In fact, it is shown that the set of all *A-dilation wavelets* of the form  $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$  (called *s-elementary wavelets*) is path-connected [10].

When discussing similar questions in the subspaces of  $L^2(\mathbb{R}^d)$ , we need to work with subspaces that are closed under *A-dilations* and translations. A closed subspace  $\mathcal{H}$  of  $L^2(\mathbb{R}^d)$  with the property that  $f(A^m t - \ell) \in \mathcal{H}$  for any  $f \in \mathcal{H}$ ,  $m \in \mathbb{Z}$  and  $\ell \in \mathbb{Z}^d$  is called a *reducing subspace* of  $L^2(\mathbb{R}^d)$ . It is shown in [4] that  $\mathcal{H}$  is a reducing subspace if and only if there exists a measurable subset *M* of  $\mathbb{R}^d$  such that  $A^t M = M$  and  $\mathcal{F}(\mathcal{H}) = L^2(\mathbb{R}^d) \cdot \chi_M$ . Subsequently, we use  $\mathcal{H}_M$  to denote a reducing subspace with the corresponding set *M* with  $A^t M = M$ . A function  $\psi \in \mathcal{H}_M$  is called an *A-dilation subspace wavelet for  $\mathcal{H}_M$*  if  $\{|\det A|^{\frac{m}{2}}\psi(A^m t - \ell) : m \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$  forms an orthonormal basis for  $\mathcal{H}_M$ . Similarly, a subset *E* of *M* is called an *A-dilation subspace wavelet*

set with respect to  $\mathcal{H}_M$  if  $\mathcal{F}^{-1}((2\pi)^{-d/2}\chi_E)$  is an  $A$ -dilation subspace wavelet for  $\mathcal{H}_M$ .

It is shown in [6] that if  $A$  is an expansive matrix,  $M$  is a set of positive measure satisfying  $M = A^t M$ , then there exists an  $A$ -dilation subspace wavelet set with respect to  $\mathcal{H}_M$  if there exists an  $A^t$ -dilation generator of  $M$  with non-empty interior, or if  $A = 2I$ . In this paper, we show that for any expansive  $A$  and any measurable set  $M$  of positive measure satisfying  $M = A^t M$ , there always exists an  $A$ -dilation subspace wavelet set with respect to  $\mathcal{H}_M$  without any additional condition. For additional information on works related to the topics covered in this paper, see [1], [2], [4], [7] and [8].

The method developed in proving our main theorem can also be used to show that bounded  $A$ -dilation wavelet sets exist for any expansive matrix  $A$ . Although this result seems to be known for a while, we have not seen a proof in the literature, so we include this in the last section of this paper.

## §2. The Main Theorem and its Proof

We begin this section by stating our main result in the following theorem.

**Theorem 1** *Let  $M$  be a set of positive measure satisfying  $M = A^t M$  where  $A$  is an expansive matrix. Then there exists an  $A$ -dilation subspace wavelet set with respect to  $\mathcal{H}_M$ .*

We need some preparations before proving Theorem 1. Two measurable sets  $E$  and  $F$  are  $2\pi\mathbb{Z}^d$ -translation congruent (or just translation congruent for short) modulo measure zero sets if there exist measurable partitions  $\{E_\ell\}_{\ell \in \mathbb{Z}^d}$  of  $E$  and  $\{F_\ell\}_{\ell \in \mathbb{Z}^d}$  of  $F$  (modulo null sets) such that  $F_\ell = E_\ell + 2\ell\pi$ . Analogously, two measurable sets  $G$  and  $H$  are  $A$ -dilation congruent if there exist measurable partitions  $\{G_n\}_{n \in \mathbb{Z}}$  of  $G$  and  $\{H_n\}_{n \in \mathbb{Z}}$  of  $H$  (modulo null sets) such that  $G_n = A^n H_n$ . A measurable set  $E$  is called an  $A$ -dilation generator of  $\mathbb{R}^d$  if  $\{A^m E : m \in \mathbb{Z}\}$  forms a partition of  $\mathbb{R}^d$  (modulo null sets), and a  $2\pi\mathbb{Z}^d$ -translation generator (translation generator) of  $\mathbb{R}^d$  if  $\{E + 2\ell\pi : \ell \in \mathbb{Z}^d\}$  is a partition of  $\mathbb{R}^d$  (modulo null sets). Observe that all the  $A$ -dilation generators of  $\mathbb{R}^d$  are  $A$ -dilation congruent to each other. Similarly all the translation generators are translation congruent to each other. So if we let  $\Omega$  be the cube

$$\Omega = \{t = (t_1, \dots, t_d) \in \mathbb{R}^d : -\pi \leq t_i < \pi, i = 1, \dots, d\},$$

then a measurable subset  $F$  of  $\mathbb{R}^d$  is a translation generator of  $\mathbb{R}^d$  if and only if  $F$  and  $\Omega$  are translation congruent. In this paper we will always use  $\Omega$  to denote this cube. For a subset  $K \subset \mathbb{R}^d$ , we use  $\mathcal{T}(K)$  to denote the set  $\cup_{\ell \in \mathbb{Z}^d} (K + 2\pi\ell)$ .

The following lemma is from [8].

**Lemma 1** *Suppose that  $G$  is a measurable subset of  $\mathbb{R}^d$  and let  $F = \Omega \cap \mathcal{T}(G)$ . Then there exists a measurable subset  $K$  of  $G$  such that  $G \setminus K$  and  $F$  are translation congruent and  $K \cap \Omega = \emptyset$ .*

Lemma 2 below can be obtained by using an argument similar to the one used in [5]. The details are left to our reader.

**Lemma 2** *Let  $M$  be a measurable subset of  $\mathbb{R}^d$  such that  $A^t M = M$ , then a measurable subset  $E$  of  $M$  is an  $A$ -dilation subspace wavelet set with respect to  $\mathcal{H}_M$  if and only if  $E$  is a translation generator of  $\mathbb{R}^d$  and an  $A^t$ -dilation generator of  $M$ .*

**Lemma 3** *Suppose that  $M$  is a set of positive measure satisfying  $M = A^t M$ . If there exists a measurable subset  $K_0$  of  $M$  satisfying*

- (i)  $\cup_{m \in \mathbb{Z}} (A^t)^m K_0 = M$ ,
- (ii)  $A^t K_0 \supset K_0$ ,
- (iii)  $\cup_{\ell \in \mathbb{Z}^d} (A^t K_0 \setminus K_0 + 2\pi\ell) = \mathbb{R}^d$  (modulo a null set),

*then there exists a subset  $K$  of  $K_0$  such that  $K$  satisfies conditions (i), (ii) above and  $A^t K \setminus K$  is a translation generator of  $\mathbb{R}^d$ .*

**Proof.** Let  $G = A^t K_0 \setminus K_0$ . By condition (iii), we have  $\mathcal{T}(G) = \mathbb{R}^d$ . Hence  $F = \Omega \cap \mathcal{T}(G) = \Omega$ . Thus, by Lemma 1, we can choose  $F_0 \subset A^t K_0 \setminus K_0$  such that  $F_0 \subset \Omega^c$  and  $(A^t K_0 \setminus K_0) \setminus F_0$  is translation congruent to  $\Omega$ . Let  $K_1 = K_0 \setminus (A^t)^{-1} F_0$ .

Since  $F_0$  and  $K_0$  are disjoint, it follows that

$$A^t K_1 = A^t K_0 \setminus F_0 \supset K_0 \setminus F_0 = K_0 \supset K_1.$$

So condition (ii) holds for  $K_1$ . Clearly

$$\begin{aligned}
A^t K_1 \setminus K_1 &= (A^t K_0 \setminus F_0) \setminus (K_0 \setminus (A^t)^{-1} F_0) \\
&\supset (A^t K_0 \setminus F_0) \setminus K_0 = (A^t K_0 \setminus K_0) \setminus F_0.
\end{aligned}$$

Therefore condition (iii) holds for  $K_1$  as well since  $(A^t K_0 \setminus K_0) \setminus F_0$  is translation congruent to  $\Omega$  hence is a translation generator of  $\mathbb{R}^d$ . Furthermore,

$$\mu(F_0) = \mu(A^t K_0 \setminus K_0) - \mu(\Omega) = (|\det A| - 1)\mu(K_0) - (2\pi)^d,$$

hence

$$\begin{aligned}
\mu(K_1) &= \mu(K_0) - \mu((A^t)^{-1} F_0) \\
&= \mu(K_0) - \left[1 - \frac{1}{|\det A|}\right]\mu(K_0) + \frac{1}{|\det A|}(2\pi)^d \\
&= \frac{1}{|\det A|}[\mu(K_0) + (2\pi)^d].
\end{aligned}$$

Now observe that there is a fixed  $\delta > 0$  such that  $K_0 \cap (\mathcal{B}(O, \delta)) \subset K_1$ , where  $(\mathcal{B}(O, \delta))$  is the ball centered at the origin with radius  $\delta$ . Also notice that if a set  $G$  satisfies conditions (i) and (ii), then any set containing  $G \cap \mathcal{B}(O, \delta)$  satisfies condition (i) as well.

We will now continue the above process to obtain a decreasing sequence  $\{K_n\}$  of measurable subsets and a sequence  $\{F_n\}$  ( $F_n \subset A^t K_n \setminus K_n$  and  $F_n \subset \Omega^c$ ) such that

- (a)  $K_0 \cap (\mathcal{B}(O, \delta)) \subset K_n$
- (b)  $A^t K_n \supset K_{n-1}$
- (c)  $(A^t K_n \setminus K_n) \setminus F_n$  is a translation generator of  $\mathbb{R}^d$ .
- (d)  $\mu(K_n) = \frac{1}{|\det A|}[\mu(K_{n-1}) + (2\pi)^d]$ .

Let  $K = \bigcap_{n=0}^{\infty} K_n$ . We can now verify that  $K$  satisfies our requirements. Due to (a) and (b), we only need to check that  $A^t K \setminus K$  is a translation generator of  $\mathbb{R}^d$ . Since  $\{K_n\}$  is decreasing,  $\mu(K) = \lim_{n \rightarrow \infty} \mu(K_n)$ . By (d), we obtain  $\mu(K) = \frac{1}{|\det A|}(\mu(K) + (2\pi)^d)$ . So  $\mu(K) = \frac{1}{|\det A| - 1}(2\pi)^d$ . Hence  $\mu(A^t K \setminus K) = (|\det A| - 1)\mu(K) = (2\pi)^d$ . Therefore, to prove that  $A^t K \setminus K$  is a translation generator of  $\mathbb{R}^d$ , it suffices to show that the sets  $\{(A^t K \setminus K) + 2\pi\ell : \ell \in \mathbb{Z}^d\}$  are disjoint. Assume, to the contrary, that there is a subset  $L$  of  $K$  with positive measure such that both  $L$  and  $L + 2\ell\pi$  are

contained in  $A^t K \setminus K$  for some  $\ell \in \mathbb{Z}^d$ , and  $L \cap (L + 2\ell\pi) \neq \emptyset$ . Note that  $A^t K \setminus K = A^t K \cap K^c = \cup_{n=0}^{\infty} A^t K \cap K_n^c$  and

$$K_0^c \subset K_1^c \subset \cdots \subset K_n^c \subset \cdots$$

where  $E^c$  denote the set complement of  $E$  in  $\mathbb{R}^d$  for any subset  $E$  of  $\mathbb{R}^d$ . By considering a subset of  $L$ , we can assume that there exists  $N > 0$  such that both  $L$  and  $L + 2\ell\pi$  are contained in  $A^t K \cap K_n^c$  for  $n \geq N$ . By the way  $F_n$  is chosen, either  $L \cap F_n$  or  $(L + 2\ell\pi) \cap F_n$  has positive measure. If  $L \cap F_n$  has positive measure, then  $(A^t)^{-1}(L \cap F_n) \cap K_{n+1} = \emptyset$ . So  $L \cap F_n \cap A^t K_{n+1} = \emptyset$ . This is a contradiction since  $L \cap F_n \subset L \subset A^t K \subset A^t K_{n+1}$ . Similarly there would be a contradiction if  $(L + 2\ell\pi) \cap F_n$  has a positive measure. Thus  $A^t K \setminus K$  must be translation disjoint.  $\square$

**Corollary 1** *Suppose that  $M$  is a measurable set of positive measure satisfying  $M = A^t M$  and  $A$  is an expansive matrix. If there is a set  $K_0$  satisfying (i), (ii) and (iii) in Lemma 3, then there exists an  $A$ -dilation subspace wavelet set with respect to  $\mathcal{H}_M$ .*

**Proof.** Let  $K$  be as in Lemma 3. Then  $A^t K \setminus K$  is a translation generator of  $\mathbb{R}^d$ . It follows that  $\cup_{n \in \mathbb{Z}} (A^t)^n (A^t K \setminus K) = M \setminus \{O\}$  since  $K$  satisfies (i) and (ii). Furthermore,  $\{(A^t)^n (A^t K \setminus K) : n \in \mathbb{Z}\}$  are disjoint sets. Thus  $A^t K \setminus K$  is an  $A^t$ -dilation generator of  $M$ . Therefore,  $A^t K \setminus K$  is an  $A$ -dilation subspace wavelet set with respect to  $\mathcal{H}_M$ .  $\square$

We are now ready to prove Theorem 1. By Corollary 1, it suffices to show that there exists a set  $K_0$  satisfying conditions (i), (ii) and (iii) in Lemma 3. Let  $F_0 \subset M$  be an  $A^t$ -dilation generator of  $M$ . Since  $M$  has positive measure,  $F_0$  must have positive measure as well. A point  $x$  is called a *Lebesgue density point* of  $F_0$  if for any  $\epsilon > 0$ , there exists a  $\delta' > 0$  such that  $\mu(F_0 \cap \mathcal{B}(x, \delta)) > (1 - \epsilon)\mu(\mathcal{B}(x, \delta))$  for any  $\delta \in (0, \delta')$ . It is a well known result (cf [9]) that modulo a null subset, all points of a set  $G$  of positive measure are Lebesgue density points of  $G$  ([9]). Let  $x$  be a Lebesgue density point of  $F_0$  and let  $\epsilon_0 \in (0, 1)$ . Then  $\mu(F_0 \cap \mathcal{B}(x, \delta_0)) > (1 - \frac{\epsilon_0}{2})\mu(\mathcal{B}(x, \delta_0))$  and  $\mu(\mathcal{B}(x, \delta_0)) < \mu(F_0)$  for some  $\delta_0 > 0$  that is small enough. Let  $\delta_1 < \delta_0$  be such that  $\mu(\mathcal{B}(x, \delta_1)) > (1 - \frac{\epsilon_0}{4})\mu(\mathcal{B}(x, \delta_0))$ . Denote the sphere of radius  $r$  centered at  $x$  by  $\mathcal{S}(x, r)$ , we have

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \{ \|(A^t)^k y_0 - (A^t)^k y_1\| : y_0 \in \mathcal{S}(x, \delta_0), y_1 \in \mathcal{S}(x, \delta_1) \} \\ = & \liminf_{k \rightarrow \infty} \{ \|(A^t)^k y\| : y \in \mathcal{S}(O, \delta_0 - \delta_1) \} = \infty \end{aligned}$$

since  $A$  is expansive. Therefore,

$$\inf\{\|(A^t)^{k_0}y_0 - (A^t)^{k_0}y_1\| : y_0 \in \mathcal{S}(x, \delta_0), y_1 \in \mathcal{S}(x, \delta_1)\} > 2\pi\sqrt{d},$$

for some  $k_0$  that is large enough. For any cube of the form  $\Omega + 2\pi\ell$ , there are three cases:

1.  $\Omega + 2\pi\ell \subset (A^t)^{k_0}(\mathcal{B}(x, \delta_0))$ ;
2.  $\Omega + 2\pi\ell \not\subset (A^t)^{k_0}(\mathcal{B}(x, \delta_0))$  but  $(\Omega + 2\pi\ell) \cap (A^t)^{k_0}(\mathcal{B}(x, \delta_0)) \neq \emptyset$ ;
3.  $(\Omega + 2\pi\ell) \cap (A^t)^{k_0}(\mathcal{B}(x, \delta_0)) = \emptyset$ .

Thus, a point of  $(A^t)^{k_0}(\mathcal{B}(x, \delta_0))$  falls in a cube satisfying either condition 1 or condition 2 above. Let  $U$  be the union of all cubes satisfying condition 1 and let  $V = (A^t)^{k_0}(\mathcal{B}(x, \delta_0)) \setminus U$ . Observe that if  $y \in V$ , then the distance between  $y$  and the boundary of  $(A^t)^{k_0}(\mathcal{B}(x, \delta_0))$  must be less than  $2\pi\sqrt{d}$ , which is the diameter of any of the cubes  $\Omega + 2\pi\ell$ . It follows that  $y \in (A^t)^{k_0}(\mathcal{B}(x, \delta_0) \setminus \mathcal{B}(x, \delta_1))$ . So we have

$$\begin{aligned} \mu(V) &< \mu((A^t)^{k_0}(\mathcal{B}(x, \delta_0) \setminus \mathcal{B}(x, \delta_1))) \\ &= |\det A|^{k_0} \mu(\mathcal{B}(x, \delta_0) \setminus \mathcal{B}(x, \delta_1)) < \frac{\epsilon_0}{4} |\det A|^{k_0} \mu(\mathcal{B}(x, \delta_0)). \end{aligned}$$

We now claim that there must exist a cube  $\Omega + 2\pi\ell_0$  for some  $\ell_0 \in \mathbb{Z}^d$  such that

$$\mu((A^t)^{k_0}(F_0) \cap (\Omega + 2\pi\ell_0)) > (1 - \epsilon_0)\mu(\Omega + 2\pi\ell_0) = (1 - \epsilon_0)(2\pi)^d.$$

For if this is not the case, then for each cube  $\Omega + 2\pi\ell$  satisfying condition 1 above, we will have  $\mu((A^t)^{k_0}(F_0) \cap (\Omega + 2\pi\ell)) \leq (1 - \epsilon_0)\mu(\Omega + 2\pi\ell)$ , and it follows that

$$\begin{aligned} \mu((A^t)^{k_0}(F_0) \cap U) &\leq (1 - \epsilon_0)\mu(U) \\ &\leq (1 - \epsilon_0)\mu((A^t)^{k_0}(\mathcal{B}(x, \delta_0))) = (1 - \epsilon_0)|\det A|^{k_0} \mu(\mathcal{B}(x, \delta_0)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\mu((A^t)^{k_0}(F_0 \cap \mathcal{B}(x, \delta_0))) \\ &= \mu((A^t)^{k_0}(F_0) \cap V) + \mu((A^t)^{k_0}(F_0) \cap U) \\ &< \frac{\epsilon_0}{4} |\det A|^{k_0} \mu(\mathcal{B}(x, \delta_0)) + (1 - \epsilon_0)|\det A|^{k_0} \mu(\mathcal{B}(x, \delta_0)) \\ &= (1 - \frac{3\epsilon_0}{4})|\det A|^{k_0} \mu(\mathcal{B}(x, \delta_0)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \mu((A^t)^{k_0}(F_0 \cap \mathcal{B}(x, \delta_0))) \\
&= |\det A|^{k_0} \mu(F_0 \cap \mathcal{B}(x, \delta_0)) \\
&> (1 - \frac{\epsilon_0}{2}) |\det A|^{k_0} \mu(\mathcal{B}(x, \delta_0)).
\end{aligned}$$

Which is a contradiction. Let  $G_0 = (A^t)^{k_0}(F_0) \cap (\Omega + 2\pi\ell_0)$  (so that  $\mu(G_0) > (1 - \epsilon_0)(2\pi)^d$ ). Since  $\mu(\mathcal{B}(x, \delta_0)) < \mu(F_0)$ , the set  $F_1 = F_0 \setminus (A^t)^{-k_0}(G_0)$  is of positive measure, so we can repeat the above for  $\epsilon_1 = \frac{\epsilon_0}{2}$  to obtain a set  $G_1$  of the form  $G_1 = (A^t)^{k_1}(F_1) \cap (\Omega + 2\pi\ell_1)$  for some  $k_1$  large enough such that  $\mu(G_1) > (1 - \epsilon_1)(2\pi)^d$  while still keeping  $F_2 = F_1 \setminus (A^t)^{-k_1}(G_1)$  with positive measure. Continuing this process, we obtain a sequence of sets  $\{G_j\}$  satisfying the following conditions:

- a. Each  $G_j$  is a subset of  $M$ ;
- b.  $\mu(G_j \cap (\Omega + 2\pi\ell_j)) > (1 - \frac{\epsilon_0}{2^j})(2\pi)^d$  for some  $\ell_j \in \mathbb{Z}^d$ ;
- c.  $(A^t)^n(G_i) \cap (A^t)^m(G_j) = \emptyset$  for any  $n, m \in \mathbb{Z}$  provided that  $i \neq j$ .

Finally, let  $G' = \cup_{j=0}^{\infty} G_j$  and  $G'' = F_0 \setminus (\cup_{k \in \mathbb{Z}} (A^t)^k(G'))$ , then  $G = G' \cup G''$  is a new  $A^t$ -dilation generator of  $M$ . We now show that  $G$  contains a translation generator of  $\mathbb{R}^d$ . Indeed, for any fixed  $\epsilon > 0$ , choose  $n \in \mathbb{Z}$  so that  $\frac{\epsilon_0}{2^n} < \epsilon$ , then for any  $\ell \in \mathbb{Z}^d$ , we have

$$\begin{aligned}
& \mu(\cup_{m \in \mathbb{Z}^d} (G + 2m\pi) \cap (\Omega + 2\ell\pi)) \\
&\geq \mu(\cup_{m \in \mathbb{Z}^d} (G_n + 2m\pi) \cap (\Omega + 2\ell\pi)) \\
&\geq \mu((G_n + 2(\ell - \ell_n)\pi) \cap (\Omega + 2\ell\pi)) \\
&= \mu((G_n \cap (\Omega + 2\ell_n\pi)) > (1 - \frac{\epsilon_0}{2^n})(2\pi)^d > (1 - \epsilon)(2\pi)^d.
\end{aligned}$$

Since  $\epsilon$  is arbitrary, we must have

$$\mu(\cup_{m \in \mathbb{Z}^d} (G + 2m\pi) \cap (\Omega + 2\ell\pi)) = (2\pi)^d$$

for any  $\ell \in \mathbb{Z}^d$ . This implies that  $\cup_{m \in \mathbb{Z}^d} (G + 2m\pi) = \mathbb{R}^d$  modulo a null set. Let  $K_0 = \cup_{m=1}^{\infty} (A^t)^{-m}G$ . It is clear that  $\cup_{m \in \mathbb{Z}^d} (A^t)^m K_0 = M$ ,  $A^t K_0 \supset K_0$ , and  $A^t K_0 \setminus K_0 = G$ , which contains a translation generator of  $\mathbb{R}^d$ . This finishes the proof of Theorem 1 in light of Lemma 3 and Corollary 1.

### §3. Existence of Bounded Wavelet Sets

The method in the proof of Lemma 3 provides us a way of constructing *bounded*  $A$ -dilation wavelet sets for any expansive matrix  $A$  and we will explore this topic in this section. We start by the following lemma.

**Lemma 4** *Let  $A$  be an expansive matrix. Then there exists a bounded set  $K_0$  satisfying the following conditions:*

- (i) *the origin  $O$  is an interior point of  $K_0$ ,*
- (ii)  *$A^t K_0 \supset K_0$ ,*
- (iii)  *$\cup_{\ell \in \mathbb{Z}^d} ((A^t K_0 \setminus K_0) + 2\pi\ell) = \mathbb{R}^d$  (modulo a null set).*

**Proof.** Since  $A$  is an expansive matrix, there is an open and bounded neighborhood  $F$  of the origin such that  $F \subset A^t F$ ,  $A^t F \setminus F$  is an  $A$ -dilation generator of  $\mathbb{R}^d$  and  $A^t F \setminus F$  has non-empty interior. It is also a known fact ([4]) that  $\cup_{m=1}^{\infty} (A^t)^{-m}(2\pi\mathbb{Z}^d)$  is dense in  $\mathbb{R}^d$  if  $A$  is expansive. Therefore, there exists an open ball  $B$  in  $A^t F \setminus F$  and a natural number  $m$ , such that  $(A^t)^{-m}(\Omega + 2\pi\ell_0) \subset B$  for some  $\ell_0 \in \mathbb{Z}^d$ . Or equivalently,  $\Omega + 2\pi\ell_0 \subset (A^t)^m(B)$ . Now let  $K_0 = A^m F$ .  $K_0$  is bounded with the origin in its interior,  $AK_0 \supset K_0$ , and  $AK_0 \setminus K_0 = A^{m+1}F \setminus A^m F \supset A^m B$ . So  $AK_0 \setminus K_0$  contains a translation generator of  $\mathbb{R}^d$  since  $\Omega + 2\pi\ell_0 \subset (A^t)^m(B)$ .  $\square$

**Theorem 2** *There exist bounded  $A$ -dilation wavelet sets for any expansive matrix  $A$ .*

**Proof.** Let  $K_0$  be a set as constructed in the proof of Lemma 4 so that it satisfies conditions (i), (ii), (iii) and is bounded. Let  $K_n$  be the sets obtained following the same procedure in the proof of Lemma 3. Let  $\delta > 0$  be small enough such that  $(A^t)^{-1}\Omega^c \cap \mathcal{B}(O, \delta) = \emptyset$  and  $\mathcal{B}(O, \delta) \subset K_0$ . We see that  $\mathcal{B}(O, \delta) \subset K_1$  by the definition of  $K_1$ . It follows that  $\mathcal{B}(O, \delta) \subset K_n$  for any  $n > 1$  by induction and the definition of  $K_n$ . Let  $K$  be as defined in the proof of Lemma 3. Then  $K$  has the following properties:

- (i)  $K$  contains  $O$  as an interior point,
- (ii)  $AK \supset K$
- (iii)  $AK \setminus K$  is a translation generator of  $\mathbb{R}^d$ ,
- (iv)  $AK \setminus K$  is bounded (since it is contained in  $AK_0$ )

Since  $K$  contains  $O$  as an interior point and  $A$  is expansive, we have that  $\cup_{n \in \mathbb{Z}} (A^t)^n K = \mathbb{R}^d$ . This implies that  $\cup_{n \in \mathbb{Z}} (A^t)^n (A^t K \setminus K) = \mathbb{R}^d \setminus \{O\}$ . Since  $\{(A^t)^n (A^t K \setminus K) : n \in \mathbb{Z}\}$  are disjoint sets,  $A^t K \setminus K$  is an  $A^t$ -dilation generator of  $\mathbb{R}^d$ . So by Lemma 2,  $AK \setminus K$  is an  $A$ -dilation wavelet set. This finishes the proof of Theorem 2.  $\square$

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