# An Adaptive Wavelet Method for Nonlinear Circuit Simulation

Dian Zhou, Wei Cai, and Wu Zhang

***Abstract*—The advance of very large scale integrated (VLSI) systems has been continuously challenging today's circuit simulators in both computational speed and stability. A novel numerical method, the fast wavelet collocation method (FWCM), was first proposed in [1] to explore a new direction of circuit simulation. The FWCM uses a totally different numerical means from the classical time-marching or frequency-domain methods and has demonstrated several superior computational properties, such as uniform error distribution and better computational stability, as compared to that provided by the conventional simulation methods.**

**The foundation for using wavelets to expand the solution of ordinary differential equations (ODE's) was laid out in [1], [2], and [10] where linear systems were computed. However, it has not been studied in detail how to effectively apply the FWCM to solving nonlinear systems. In this sequel paper, we explore the iterative and adaptive schemes which extend the FWCM to nonlinear systems. The proposed adaptive procedures mainly address the method of linearization of nonlinear terms after the unknown vector function is expanded into wavelet basis functions. We implemented two different adaptive schemes, multilevel adaptive and multiinterval adaptive, and evaluated their advantages and disadvantages. It is shown that the FWCM can handle nonlinear systems very efficiently with an accuracy as high as $O(h^4)$ for the solution and fast mapping between the values of the function and their wavelet expansion coefficients in, at most, $O(N \log N)$ operations, where $h$ is the discrete time-interval length and $N$ is the total number of collocation points. Furthermore, the derivatives of the unknown function can be calculated with an accuracy of $O(h^3)$ in $O(N \log N)$ operations. Numerical results are presented which match well with the SPICE simulation.**

***Index Terms*—Circuit simulation, VLSI, wavelet method.**

## I. Introduction

The time-marching method is the most popular numerical approach in VLSI circuit simulation because circuit nonlinear properties can be well handled in the time domain [3]. However, the method suffers from inefficiency in treating singularities, which often develop in high-speed circuits. It also has the problem of nonuniform error distribution that may cause a phase shifting phenomenon of the simulated signal [4]. The importance of uniform approximation was explored in [1] and [10] where a fast wavelet collocation method (FWCM) was introduced to approximate solutions of ordinary differential equations with a uniform error distribution.Theoretical analysis and numerical results of applying FWCM to the linear circuit indicated that the FWCM is indeed a robust algorithm, by which the circuit simulation can be performed very efficiently. The purpose of this paper is to study the potential of extending the FWCM to solving nonlinear systems and to develop the corresponding adaptive schemes.

The paper is organized as follows. Section II first outlines the general format of the FWCM and then extends it to solving nonlinear systems. It presents the discretization method and iterative schemes. Section III demonstrates the proposed algorithm with numerical experiments of nonlinear circuits. Section IV comments on the further research. To ensure that the paper is self-contained and for readers who are interested in the mathematical completeness of the FWCM, the Appendix describes the background of the FWCM, including the construction of approximation subspaces, definition of basis function sets, wavelet expansion of any function having second-order continuous generalized derivatives, and the transform between expansion coefficients of the function and its values.

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## II. The FWCM for Nonlinear Problems

This section deals with the application of the FWCM to nonlinear ordinary differential equations. Specifically, we shall describe the entire numerical procedure of the FWCM, including the expansion of the unknown solution, the discretization of the nonlinear system, the iterative scheme, and adaptive techniques.

### A. Function Expansion

Let $H^2(I)$ be a Sobolev space which basically contains functions with square integrable second derivatives [5]. We introduce a basis function set of approximation subspace $V_{bJ} \subset H^2(I)$ for a given integer $J \geq 0$ and a fixed interval $I = [0, L]$ with $L > 4$ [2], [10]

$$\begin{aligned} V_{bJ} &= \{\eta_1(t), \eta_2(t), \eta_2(L-t), \eta_1(L-t) \\ &\quad \varphi_{0,-1}(t), \varphi_{0,0}(t), \ldots, \varphi_{0,L-4}(t), \varphi_{0,L-3}(L-t) \\ &\quad \psi_{0,-1}(t), \psi_{0,0}(t), \psi_{0,1}(t), \ldots, \psi_{0,n_0-3}(t), \psi_{0,n_0-2}(t) \\ &\quad \cdots\cdots\cdots\cdots\cdots \\ &\quad \psi_{j,-1}(t), \psi_{j,0}(t), \psi_{j,1}(t), \ldots, \psi_{j,n_j-3}(t), \psi_{j,n_j-2}(t) \\ &\quad \cdots\cdots\cdots\cdots\cdots \\ &\quad \psi_{J-1,-1}(t), \psi_{J-1,0}(t), \psi_{J-1,1}(t), \ldots, \psi_{J-1,n_{J-1}-3}(t), \\ &\quad \psi_{J-1,n_{J-1}-2}(t)\} \end{aligned} \tag{2.1}$$

where $n_j = 2^j L$. We further define a set of wavelet subspaces

$$V_b = \{\eta_1(t), \eta_2(t), \eta_2(L-t), \eta_1(L-t)\} \tag{2.2}$$

$$V_0 = \text{span}\{\varphi_{0,-1}(t), \varphi_{0,k}(t), 0 \le k \le L-4 \quad \varphi_{0,L-3}(L-t)\} \tag{2.3}$$

$$W_j = \text{span}\{\psi_{j,k}(t), -1 \le k \le n_j - 2\}, \qquad 0 \le j \le J-1, \tag{2.4}$$

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}, \qquad 0 \le j \le J-1 \tag{2.5}$$

$$V_{bj} = V_b \cup V_j \tag{2.6}$$

where the notation $V \oplus W$ stands for the direct sum and $\text{span}\{f_1, f_2, \ldots, f_n\}$ represents a function set formed by all linear combinations of the functions $f_1, f_2, \ldots, f_n$. The subspaces defined in (2.2)–(2.6) form a wavelet decomposition of the Sobolev space $H^2(I)$ [2], [10]. The orthogonal wavelet proposed by Daubechies was originally designed for $L^2(R)$, with $R$ being the whole real line [14]. Hence, for any function $x(t) \in H^2(I)$, we can expand it in a series of the basis function system defined by (2.1) and consider its interpolation at some interior knots, called collocation points, defined in (2.12) and (2.13) below. We write the expansion of the function $x(t)$ as

$$x_J(t) = I_{V_{b0}} x(t) + \sum_{j=0}^{J-1} I_{W_j} x(t) \tag{2.7}$$

where

$$\begin{aligned} I_{Vb0} x(t) &= \hat{x}_{-1,-3}\eta_1(t) + \hat{x}_{-1,-2}\eta_2(t) + \hat{x}_{-1,-1}\varphi_b(t) \\ &\quad + \sum_{k=0}^{L-4} \hat{x}_{-1,k}\varphi_k(t) + \hat{x}_{-1,L-3}\varphi_b(L-t) \\ &\quad + \hat{x}_{-1,L-2}\eta_2(L-t) + \hat{x}_{-1,L-1}\eta_1(L-t) \\ &= x_{-1}(t) \in V_{b0} \end{aligned} \tag{2.8}$$

and

$$I_{W_j} x(t) = \sum_{k=-1}^{n_j-2} \hat{x}_{j,k}\psi_{j,k}(t) = x_j(t) \in W_j, \quad 0 \le j \le J-1. \tag{2.9}$$

We denote expansion coefficients as a vector

$$\begin{aligned} \hat{x}_J = (&\hat{x}_{-1,-3}, \hat{x}_{-1,-2}, \ldots, \hat{x}_{-1,k}, \ldots, \hat{x}_{-1,L-2}, \hat{x}_{-1,L-1} \\ &\hat{x}_{0,-1}, \hat{x}_{0,0}, \ldots, \hat{x}_{0,k}, \ldots, \hat{x}_{0,n_o-3}, \hat{x}_{0,n_o-2}; \ldots\ldots\ldots \\ &\hat{x}_{J-1,-1}, \hat{x}_{J-1,0}, \ldots, \hat{x}_{J-1,k}, \ldots, \hat{x}_{J-1,n_{j-1}-3} \\ &\hat{x}_{J-1,n_{J-1}-2}) \end{aligned} \tag{2.10}$$

which will be determined by satisfying interpolating conditions at collocation points $t_k^{(-1)}$, $1 \le k \le L+3$ and $t_k^{(j)}$, $-1 \le k \le n_j - 2$, $j \ge 0$, i.e.,

$$\begin{cases} I_J x\big(t_k^{(-1)}\big) = x\big(t_k^{(-1)}\big), & 1 \le k \le L+3 \\ I_J x\big(t_k^{(j)}\big) = x\big(t_k^{(j)}\big), & j \ge 0; \quad -1 \le k \le n_j - 2, \\ & 0 \le j \le J-1. \end{cases} \tag{2.11}$$

The following collocation points are chosen for $V_{b0}$ and $W_j, j \ge 0$, respectively

$$\begin{aligned} &t_1^{(-1)} = 0, \quad t_2^{(-1)} = \frac{1}{2}, \quad t_k^{(-1)} = k-2, \quad 3 \le k \le L+1 \\ &t_{L+2}^{(-1)} = L - \frac{1}{2}, \quad t_{L+3}^{(-1)} = L \end{aligned} \tag{2.12}$$

$$\begin{aligned} &t_{-1}^{(j)} = \frac{1}{2^{j+2}}, \quad t_k^{(j)} = \frac{k+1.5}{2^j}, \qquad 0 \le k \le n_j - 3 \\ &t_{n_j-2}^{(j)} = L - \frac{1}{2^{j+2}}. \end{aligned} \tag{2.13}$$

Note that the total number of collocation points is $N = 2^J L + 3$, which equals exactly the number of the basis functions defined in (2.1) and the number of the expansion coefficients in (2.7)–(2.9). Consequently, we can uniquely determine the coefficients by the above interpolating conditions.

### *B. Discretization of Nonlinear System*

Consider the following nonlinear ordinary differential equations:

$$\frac{dx}{dt} = f(t, x) \tag{2.14a}$$

$$x(0) = x_0 \tag{2.14b}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is an unknown vector function and $f(t, x)$ is a given nonlinear vector function. We expand the unknown function $x(t)$ for a given integer $J \ge 0$ and obtain

$$\begin{aligned} I_J x(t) = I_{V_{b0}} x(t) + \sum_{j=0}^{J-1} I_{W_j} x(t) = x_{-1}(t) \\ + \sum_{j=0}^{J-1} x_j(t) = x_J(t) \end{aligned} \tag{2.15}$$

where $x_{-1}(t)$ and $x_j(t)$, $0 \le j \le J-1$ have the same forms as those in (2.8) and (2.9), respectively. Interpolating $I_J x(t)$ in (2.15) at the collocation points $\{t_k^{(j)}\}, j \ge -1$ in (2.13) and substituting the expressions (2.15) into (2.14), we obtain the following nonlinear discrete algebraic system:

$$A\hat{x} = f(\hat{x}) \tag{2.16}$$

where $\mathbf{A}$ is an $N$-by-$N$ invertible derivative matrix $\hat{x} = \hat{x}_J$ given in (2.10) and $f(\hat{x})$ is the vector of the values of $f(\hat{x})$ at all collocation points ordered in the same way as $\hat{x}$. For a given integer $J \ge 0$, $\mathbf{A}$ is a constant matrix. Elements of $\mathbf{A}$ are given by calculating derivative values of basis functions at each collocation point (for details on $\mathbf{A}$, see [10]). Solving (2.16), we obtain solution $\hat{x}$ of the expansion coefficients. From a fast discrete wavelet transform (DWT) technique, we can effectively construct the approximation solution $x_J(t)$ of the ordinary differential equations (2.14) from the expansion coefficients $\hat{x}$ [2].

### *C. Iterative Schemes*

Since the system (2.16) is nonlinear, it needs to be solved by some kind of iterative method. We study the following two iterative schemes.

*Scheme 1*: [Using (2.16) directly.]

We rewrite (2.16) as follows:

$$\hat{x} = A^{-1} f(\hat{x}) \tag{2.17a}$$

and then have the iterative scheme

$$\hat{x}^{(k+1)} = A^{-1} f\big(\hat{x}^{(k)}\big). \tag{2.17b}$$

The following is a formal computation procedure.

Step 1) Choose initial iterative vector $\hat{x}^{(0)} = \hat{x}_0$ and give a tolerance $\varepsilon > 0$.

Step 2) Find $\hat{x}^{(k+1)}$ from (2.17) where $k = 0$ for the first iteration.

Step 3) Compute error $E$ where

$$E = \big\|\hat{x}^{(k+1)} - \hat{x}^{(k)}\big\| / \big\|\hat{x}^{(k)}\big\|. \tag{2.18}$$

Stop computing if $E < \varepsilon$. Otherwise, go to Step 2.

The scheme described in (2.17) is a fixed-point scheme which corresponds to $\hat{x} = T\hat{x}$ where $T$ is an operator determined by matrix $\mathbf{A}$ and function $f$.

*Scheme 2*: [Using an equivalent form of (2.16).]

We rewrite (2.16) as the following equivalent form:

$$\hat{x} = \hat{x} - \rho(A\hat{x} - f(\hat{x})). \tag{2.19a}$$

Thus, we obtain the corresponding iterative scheme as follows:

$$\begin{aligned} \hat{x}^{(k+1)} &= \hat{x}^{(k)} - \rho\big(A\hat{x}^{(k)} - f\big(\hat{x}^{(k)}\big)\big) \\ &= \hat{x}^{(k)} - \rho\left(\frac{dx}{dt}^{(k)} - f\big(t, x^{(k)}\big)\right) \end{aligned} \tag{2.19b}$$

where $0 < \rho < 1$ is a relaxation factor and determined by the condition number of the matrix $(A - \frac{\partial f}{\partial \hat{x}})$ where $-\frac{\partial f}{\partial \hat{x}}$ is the Jacobian matrix of vector $f(\hat{x})$ with respect to vector $\hat{x}$. Equations (2.19) is also a fixed-point iterative scheme and converges if the $\rho$ value chosen is small enough [13]. The iterative procedure is as follows.

Step 1) Choose initial iterative vector $\hat{x}^{(0)} = \hat{x}_0$ and give a small real number $\varepsilon > 0$.

Step 2) Calculate $A\hat{x}^{(k)}$ and $f(\hat{x}^{(k)})$ or $\frac{dx}{dt}$ and $f(t, x)$. Let $k = 0$ for the first iteration.

Step 3) Find $\hat{x}^{(k+1)}$ from (2.19).

Step 4) Compute error $R$ defined by (2.18). Stop computing if $R < \varepsilon$. Otherwise, go to Step 2.

Some suggestions about the above iterative schemes will be given in Section III. In general, we would rather use Scheme 2, which is a convergent scheme, if the $\rho$ value chosen is small enough. Scheme 1 may diverge because it is possible that the Scheme (2.17b) may not produce a contraction operator. In fact, unlike Scheme 2, iterations in Scheme 1 cannot be controlled during iterative computation. We shall thereafter consider iterative Scheme 2 only.

Finally, we discuss how to choose initial starting value $\hat{x}^{(0)}$ with which the discrete system (2.16) is solved iteratively. Since we solve the nonlinear system (2.16) in the whole interval $I$ simultaneously, we must provide initial values along the interval $I$. Two kinds of initial starting values are considered in the present paper. One uses the initial value (2.14b) of the ordinary differential equations. That is, we assume that $x(t) = x_0$ for all collocation points. The other depends upon the input of the system studied.

### D. Adaptive Techniques

One of the main advantages of the FWCM is that adaptive procedures can be used [1], [2]. Using adaptive techniques, we can improve the computational efficiency significantly. In the following we consider two types of adaptive procedures.

*1) Multilevel Adaptive:* As a result of the wavelet space decomposition property (2.5), higher resolution of the solution can be obtained by increasing the level $J$ of the wavelet expansion in (2.7). More importantly, because of compact support of the wavelet basis, not all wavelet basis functions $\psi_{j,k}(t)$ are needed in higher wavelet spaces $W_{J'}$ in order to improve the resolution of the singularities in the solution. Moreover, it is well known [2], [14] that the wavelet coefficients of an approximation to a function reflect the singularity of the function. Therefore, by thresholding the wavelet coefficients within a given error tolerance $\varepsilon$ we will only retain those wavelet basis functions which have contributed significantly to accuracy. Thus, the magnitude of the wavelet coefficient in $W_J$ will indicate whether a refinement, by increasing the wavelet space level, is needed or not. For example, given error tolerance $\varepsilon$, if

$$\max |\hat{x}_{J,k}| > \varepsilon \tag{2.20}$$

then we can increase the number of wavelet subspaces $W_j$ in the expansion for the numerical solution $x_J(t) = I_J x(t)$ in (2.3), say, up to $W_{J'}, J' > J$. Let $J' > J$ and define a solution vector

$$x_{J'} = I_{J'} x = \big(x^{(-1)}, x^{(0)}, \ldots, x^{(J-1)}, x^{(J)}, \ldots, x^{(J'-1)}\big) \tag{2.21}$$

where $x^{(j)} = \{I_{J'} x(t_k^{(j)})\}_{k=-1}^{n_j-2}, J \le j \le J' - 1$. Next, $x_{J'} \in V_b \cup V_0 \oplus W_0 \oplus \cdots \oplus W_{J'-1}$ will be the new numerical solution which yields a better approximation of the exact solution of (2.14).

Again, only basis functions, thus their coefficients near the singularities (which could be determined by the magnitude of the wavelet coefficients on level $W_J$), shall be included in the higher wavelet spaces $W_{J'}$.

Another improvement of the convergence is to apply the iterative scheme (2.19a) to find the difference of $x(t) - x_J(t)$ where $x_J(t)$ is the previous solution of (2.15), using wavelet space up to $W_J$ and $x(t)$; the exact solution of (2.14). Thus, a wavelet expansion up to level $W_{J'}$ in (2.21) will be applied to $x(t) - x_J(t)$. Such an approach is very close to the defect correction method [11] in the context of wavelet approximations.

*2) Multiinterval Adaptive:* Another improvement of accuracy is to use the idea of multidomain methods commonly used in computational fluid dynamics [12] where the physical domain is decomposed into subdomains. In our situation, the whole time interval will be subdivided into subintervals, hence the term multiinterval adaptive. Within each subinterval, we can apply any numerical technique (i.e. iterative Schemes 1 or 2, multilevel adaptive). Our experience with the singular solution is that the combination of multiinterval and multilevel adaptive techniques give the best performance in terms of rate of convergence and accuracy, as shown in the numerical results given below.

## III. Numerical Experiment

For linear circuit simulation, [1] compared the FWCM with the time matching methods, including Euler, Runge–Kutta, and those used in SPICE. The results demonstrated that the FWCM has a superior property in improving computational stability, overcoming the phase shifting phenomenon, and providing uniform approximation. In the following we show the performance of the FWCM for nonlinear circuit simulation and compare it with SPICE. Note that the interval $I$ in the previous discussion is a standard interval $I = [0, L]$ where $L$ is a positive integer. For a real physical time interval $I_p = [t_0, t_1]$, $t_0 \geq 0$ and $t_1 \geq t_0$ we shall first map it to the standard interval and then apply the FWCM.

### *A. Nonlinear Circuit*

A CMOS inverter is used here as a representative example of the nonlinear circuit for its simplicity [Fig. 1(a)]. Let $v(t)$, the output voltage of the inverter, be the unknown function. The nonlinear ODE describing the circuit behavior is

$$\begin{cases} \dfrac{dv(t)}{dt} = f(t, v(t)) \\ v(t_0) = v_0 \end{cases} \tag{3.1}$$

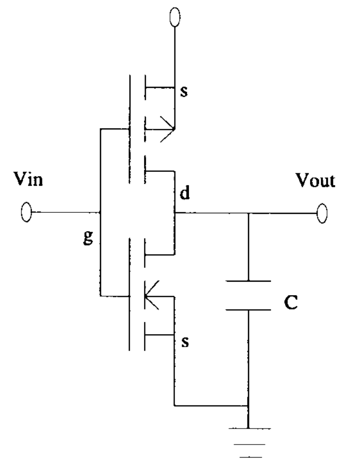
where

$$f(t, v(t)) = \frac{I_{\mathrm{ds},P\mathrm{mos}}(t) - I_{\mathrm{ds},N\mathrm{mos}}(t)}{C} \tag{3.2}$$

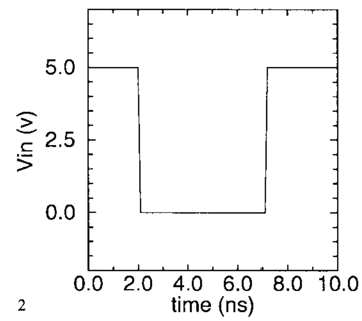
and (3.3), at the bottom of this page, in which $v_{\mathrm{th}}$, $K_p$, $X_{jl}$, and $\lambda$ are the transistor's model parameters [8], [9] and $W$ and $l$ are, respectively, the channel width and length. The input voltage $v_{\mathrm{gs}}$ is assumed to have the form shown in Fig. 1(b). Initially, the load capacitor $C$ is discharged. The FWCM and numerical techniques described in the previous sections were implemented in C++, and the computation was performed on a Sun Sparc 20 workstation. We now analyze the effectiveness of the proposed numerical techniques and choose the computation parameters involved.

*1) Selection of Parameters $J$ and $L$:* To use the FWCM, we must first choose level parameter $J$ of approximation space $V_J, J \geq 0$ and length parameter $L$ of the standard interval. The reasonable values of $J$ and $L$ are important in increasing the simulation efficiency. In general, low values of $J$ and $L$ may result in convergence difficulty and, on the other hand, overly large values may increase computation cost. To highlight this point numerically, we show in Tables I and II the choice of parameters $J$ and $L$ and the computation data, such as central processor unit (CPU) time, number of iterations $n$, and errors $R1$ and $R2$ for solving the nonlinear equations (3.1)–(3.3) where $R1$ and $R2$ are, respectively, the error of nonlinear iterations and the error between the present results and the SPICE's results in the sense of (2.18).

From Tables I and II, we see that parameters $J$ and $L$ affect both the number $n$ of nonlinear iterations and the total CPU time. The values of error $R2$ indicate that there is an optimal match between values $J$ and $L$ for numerical computation where $J = 2$ and $L = 20$ in the present case. Table III gives




(a)




(b)

Fig. 1. (a) A CMOS inverter. (b) The input signal.

TABLE I
Numerical Results for $J = 2$

| L | N | R1 | R2 | n | CPU time (second) |
|---|---|---|---|---|---|
| 5 | 23 | 9.82 e-4 | 3.60 e0 | 8 | 1.48 |
| 10 | 43 | 7.27 e-4 | 1.05 e-1 | 14 | 7.58 |
| 20 | 83 | 4.21 e-4 | 1.81 e-3 | 12 | 19.97 |
| 40 | 163 | 2.96 e-2 | 1.31 e-2 | 20 | 123.6 |

$$I_{\mathrm{ds}} = \begin{cases} K_p \dfrac{W}{l - 2X_{jl}} \left( v_{\mathrm{gs}} - v_{\mathrm{th}} - \dfrac{v_{\mathrm{ds}}}{2} \right) v_{\mathrm{ds}} (1 + \lambda v_{\mathrm{ds}}), & v_{\mathrm{gs}} > v_{\mathrm{th}} \text{ and } v_{\mathrm{ds}} < v_{\mathrm{gs}} - v_{\mathrm{th}} \\ \dfrac{K_p}{2} \dfrac{W}{l - 2X_{jl}} (v_{\mathrm{gs}} - v_{\mathrm{th}})^2 (1 + \lambda v_{\mathrm{ds}}), & v_{\mathrm{gs}} > v_{\mathrm{th}} \text{ and } v_{\mathrm{ds}} > v_{\mathrm{gs}} - v_{\mathrm{th}} \\ 0, & \text{others} \end{cases} \tag{3.3}$$

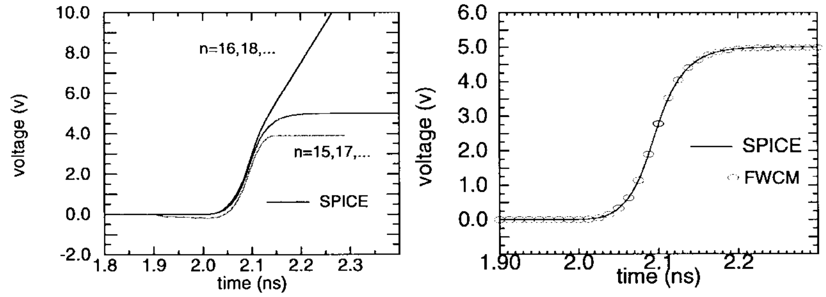



Fig. 2. Results of using iterative Scheme 1.

TABLE II
NUMERICAL RESULTS FOR $L = 20$

| J | N | R1 | R2 | n | CPU time (second) |
|---|---|---|---|---|---|
| 0 | 23 | 3.92 e-3 | 5.55 e-3 | 20 | 1.32 |
| 1 | 43 | 5.84 e-4 | 2.69 e-3 | 12 | 4.39 |
| 2 | 83 | 4.21 e-4 | 1.81 e-3 | 12 | 19.97 |
| 3 | 163 | 6.71 e-4 | 2.47 e-3 | 10 | 86.74 |

TABLE III
NUMERICAL RESULTS FOR $N = 83$

| L | J | R1 | R2 | n | CPU time (second) |
|---|---|---|---|---|---|
| 5 | 4 | 1.11 e-4 | 6.26 e-1 | 8 | 27.74 |
| 10 | 3 | 9.07 e-4 | 1.29 e-1 | 15 | 42.25 |
| 20 | 2 | 4.21 e-4 | 1.81 e-3 | 12 | 19.97 |
| 40 | 1 | 9.21 e-2 | 4.60 e-2 | 20 | 25.41 |

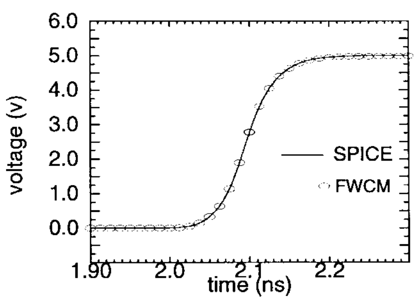



Fig. 3. Results of using iterative Scheme 2.

the results calculated under the condition of a fixed number $N$ of collocation points. $J = 2$ and $L = 20$ are still the best match in all combinations of $J$ and $L$. However, the combination of $J = 4$ and $L = 5$ also gives comparable results and may provide more potential for adaptivity as, at a higher level of wavelet spaces, more wavelet basis functions could be dropped if they are not close to the singularity in the approximated solution.

*2) Iterative Schemes:* We apply two different iterative schemes given by (2.17) and (2.19) to solve nonlinear equation (3.1). Two sets of results are shown in Figs. 2 and 3 where the approximation-level parameter $J = 3$ and the standard interval length $L = 40$. As indicated in Section III that Scheme 1 may diverge for some cases, Fig. 2 demonstrates such an incident

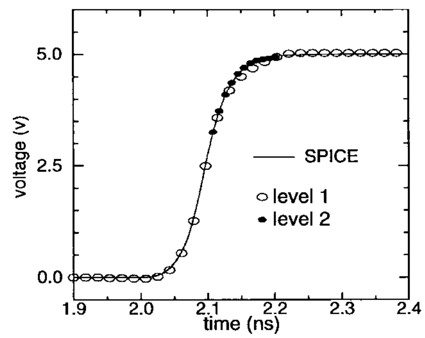



Fig. 4. Results from multilevel adaptive.

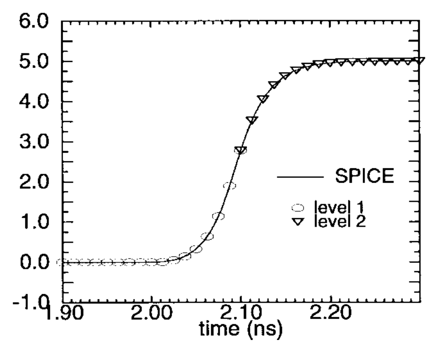



Fig. 5. Results from multiinterval adaptive.

with the simulation result oscillating when iterative Scheme 1 is used. The results calculated by using iterative Scheme 2 match very well with the SPICE simulation result and, hence, it is recommended in this paper.

*3) Adaptive Techniques:* According to the analysis in Section III, we use two different adaptive techniques in the present circuit simulation, i.e., the multilevel adaptive and multiinterval adaptive. Fig. 4 gives the results obtained by the multilevel adaptive, where $L = 20$ and $J = 0, 1$. Level 1 in Fig. 4 stands for the results of $J = 0$. Level 2 represents the results of $J = 1$ where only the function difference between two subspace levels is considered during computation. Fig. 5 shows the results obtained by using multiinterval adaptive. The interval [1.9., 2.3] is divided into two subintervals, [1.9, 2.1] and [2.1, 2.3]. In Fig. 5, circles indicate the results obtained on the interval [1.9, 2.1] and triangles on interval [2.1, 2.3]. Clearly, both adaptive techniques generated satisfactory results. Fig. 6 shows the results of combining multilevel and multiinterval adaptive techniques. The whole interval [0, 10]

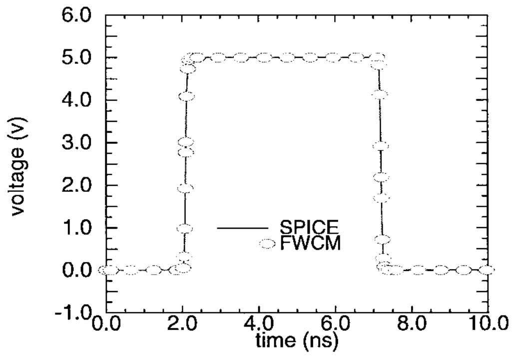



Fig. 6. Results from the combination of two techniques above.

is divided into five subintervals. On each of the subintervals, multilevel adaptive (up to $J = 3$) is applied.

Through the numerical experiment we also find that the selection of the iterative initial value of nonlinear discrete systems is very important. In general, whether initial value is close to the solution not only affects the convergence rate, but also may result in divergence. With both parameters $L$ and $J$ given, we always select the computational results of adjacent lower level and previous standard interval length as initial values of the higher level and/or larger interval. The numerical experiment implies that the convergence rate is then significantly improved. Note that it is very difficult to obtain a convergent solution with the whole time interval considered simultaneously. We suggest that the multiinterval adaptive should be used to divide the large physical intervals into smaller ones.

## IV. Conclusion

The FWCM discussed in this paper has successfully produced solutions of nonlinear systems. The proposed techniques for treating the boundary conditions and the adaptive procedures make it possible to improve both the computational efficiency and numerical accuracy. The property that high-level wavelet basis functions can be obtained by translating and dilating the lower basis functions makes it very simple to implement the computation code.

We conclude by mentioning a few areas requiring further work. First, we are going to do more numerical tests for various practical circuit systems in order to make the FWCM a practical simulation method. Special emphasis will be placed on making numerical results that meet the theoretical analysis. Second, we will construct more powerful iterative schemes and establish some criteria for using adaptive procedures for different types of nonlinear problems, especially large nonlinear systems. Finally, we should mention that an available tool for simulating practical circuit systems on a relatively large scale will be developed, based on the proposed FWCM.

## Appendix

### A. Approximation Function Spaces and Basis Functions

For any numerical computation in which function expansion is considered, approximation function spaces and basis functions always play an important role in the related numerical method. We now discuss those used in the FWCM.

We first introduce the following function spaces:

$$V_0 = \text{span}\{\varphi_{0,-1}(t), \varphi_{0,k}(t), \qquad 0 \le k \le L-4, \varphi_{0,L-3}(L-t)\} \tag{A.1}$$

$$W_j = \text{span}\{\psi_{j,k}(t), -1 \le k \le n_j - 2\}, \quad j \ge 0 \tag{A.2}$$

where the definitions of functions $\varphi_{0,-1}(t)$, $\varphi_{0,k}(t)$, $0 \le k \le L-4, \varphi_{0,L-3}(L-t)$ and $\psi_{j,k}(t), -1 \le k \le n_j - 2, j \ge 0$ are given below. $\text{span}\{f_1, f_2, \ldots, f_n\}$ is a function set formed by all linear combinations of the functions $f_1, f_2, \ldots, f_n$. From definitions (A.1) and (A.2), we obtain the following results:

$$V_0 \subset V_1 \subset V_2 \subset \ldots \tag{A.3}$$

where $V_j = V_{j-1} + W_{j-1}, j \ge 1$.

It can be proved that

(i)

$$V_{j+1} = V_j \oplus W_j, \qquad \text{for } j \in Z^+.$$

Therefore

(ii)

$$W_j \perp W_{j+1}, \qquad j \in Z^+$$

(iii)

$$H_0^2(I) = V_0 \oplus_{j \in Z^+} W_j$$

where $Z^+$ is a positive integer set and the orthogonality is defined in [10].

From the above properties, any function $x(t) \in H_0^2(I)$ can be approximated as closely as needed by a function $x_j(t) \in V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$ for a sufficiently large $j$. In this case, the approximation relates to a homogeneous problem, i.e., $x(0) = x(L) = 0$.

For any function $x(t) \in H^2(I)$, $x(0) \ne 0$, $x(L) \ne 0$, we introduce a set of boundary functions

$$V_b = \{\eta_1(t), \eta_2(t), \eta_2(L-t), \eta_1(L-t)\} \tag{A.4}$$

where the definitions of functions $\eta_1(t), \eta_2(t), \eta_2(L-t), \eta_1(L-t)$ are given below again. If we use $V_b$ to approximate boundary values of $x(t) \in H^2(I)$, then $x_j(t) \in V_b \cup V_j$ will still have an approximation to $x(t)$ of order $O(2^{-4j})$ for a given $j \ge 0$. The present approximation will be used to treat a nonhomogeneous problem. For simplicity, we let $V_{bj} = V_b \cup V_j, j \ge 0$.

We first introduce approximation subspace $V_{bJ} \subset H^2(I)$ for a given integer $J \ge 0$ and a fixed interval $I = [0, L]$, consisting of scaling functions and wavelet functions

$$\begin{aligned} V_{bJ} = \{&\eta_1(t), \eta_2(t), \eta_2(L-t), \eta_1(L-t);\\ &\varphi_{0,-1}(t), \varphi_{0,0}(t), \ldots, \varphi_{0,L-4}(t), \varphi_{0,L-3}(L-t)\\ &\psi_{0,-1}(t), \psi_{0,0}(t), \psi_{0,1}(t), \ldots, \psi_{0,n_0-3}(t), \psi_{0,n_0-2}(t)\\ &\cdots\cdots\cdots\cdots\\ &\psi_{j,-1}(t), \psi_{j,0}(t), \psi_{j,1}(t), \ldots, \psi_{j,n_j-3}(t), \psi_{j,n_j-2}(t)\\ &\cdots\cdots\cdots\cdots \end{aligned}$$

$$\psi_{J-1,-1}(t), \psi_{J-1,0}(t), \psi_{J-1,1}(t), \dots \psi_{J-1,n_{J-1}-3}(t), \psi_{J-1,n_{J-1}-2}(t)\}. \tag{A.5}$$

The scaling functions in (2.1) are boundary scaling functions

$$\varphi_{0,-1}(t) = \varphi_b(t) = \frac{3}{2}t_+^2 - \frac{11}{12}t_+^3 + \frac{3}{2}(t-1)_+^3 - \frac{3}{4}(t-2)_+^3 + \frac{1}{6}(x-3)_+^3 \tag{A.6a}$$

$$\varphi_{0,L-3}(L-t) = \varphi_b(L-t) \tag{A.6b}$$

and the interior scaling functions

$$\varphi_{0,k}(t) = \varphi(t-k), \quad 0 \le k \le L-4 \tag{A.6c}$$

where

$$\varphi(t) = N_4(t) = \frac{1}{6}\sum_{l=0}^{4}\binom{4}{l}(-1)^l(t-l)_+^3 \qquad t_+^n = \begin{cases} t^n, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases} \tag{A.6d}$$

and $N_4(t)$ is the fourth-order B spline [6].

The functions $\eta_1(t)$, $\eta_2(t)$ are used to handle nonhomogeneity of the boundary data

$$\eta_1(t) = (1-t)_+^3 \tag{A.7a}$$

$$\eta_2(t) = 2t_+ - 3t_+^2 + \frac{7}{6}t_+^3 - \frac{4}{3}(t-1)_+^3 + \frac{1}{6}(t-2)_+^3. \tag{A.7b}$$

The boundary wavelet functions are

$$\psi_{j,-1}(t) = \psi_{b0}(2^j t), \quad \psi_{j,0}(t) = \psi_{b1}(2^j t) \tag{A.8a}$$

$$\psi_{j,n_j-3}(t) = \psi_{b1}(2^j(L-t)), \quad \psi_{j,n_j-2}(t) = \psi_{b0}(2^j(L-t)) \tag{A.8b}$$

where

$$\psi_{b0}(t) = -\frac{56}{99}(14\psi_{0,-2}(t) + \psi_{0,-1}(t)) \tag{A.8c}$$

$$\psi_{b1}(t) = \frac{182}{181}\left(\psi(t) + \frac{1}{13}(\psi_{0,-1}(t) + \psi_{0,-2}(t))\right) \tag{A.8d}$$

$$\psi_{0,-1}(t) = \psi(t+1), \quad \psi_{0,-2}(t) = \psi(t+2) \tag{A.8e}$$

and the interior wavelet functions are

$$\psi_{j,k}(t) = \psi(2^j t - k), \quad k = 1, \dots, n_j - 4 \tag{A.8f}$$

where $n_j = 2^j L$ and

$$\psi(t) = -\frac{3}{7}\varphi(2t) + \frac{12}{7}\varphi(2t-1) - \frac{3}{7}\varphi(2t-2). \tag{A.8g}$$

the total number of basis functions in (A.5) is $N = L + 3 + \sum_{j=0}^{J-1} 2^J L = 2^J L + 3$.

We now discuss the interpolation of a function $x(t) \in H^2(I)$ at boundary points 0 and $L$ by means of the boundary interpolating functions $\eta_1(t), \eta_2(t)$ and $\eta_2(L-t), \eta_1(L-t)$ defined in (A.7). Based on the Sobolev embedding theorem [5], it is true that $x(t) \in C^1(I)$. We can define the following interpolating spline $I_{b,j}x(t), j \ge 0$:

$$I_{b,j}x(t) = \alpha_1\eta_1(2^j t) + \alpha_2\eta_2(2^j t) + \alpha_3\eta_1(2^j(L-t)) + \alpha_4\eta_2((2^j(L-t))) \tag{A.9}$$

where the coefficients $\alpha_i, i = 1,2,3,4$ are determined by certain interpolating conditions. Since spline $I_{b,j}x(t)$ is expected to approximate the nonhomogeneities of the function $x(t) \in H^2(I)$ at the boundaries, these interpolating conditions are also called the end conditions. In this paper, in particular, we use the following so-called not-a-knot conditions, which amounts to requiring that the spline $I_{b,j}x(t)$ agree with function $x(t)$ at one additional point near each boundary. Hence, we have the following equations for $\alpha_i, i = 1,2,3,4$:

$$\begin{aligned} I_{b,j}x(0) &= x(0), \quad I_{b,j}x(L) = x(L) \\ I_{b,j}x(\tau_1) &= x(\tau_1), \quad I_{b,j}x(\tau_2) = x(\tau_2). \end{aligned} \tag{A.10}$$

In our case, by choosing $\tau_1 = \frac{1}{2^{j+1}}$, $\tau_2 = L - \frac{1}{2^{j+1}}$, we have

$$\begin{aligned} \alpha_1 &= x(0), \qquad \alpha_2 = 6x(\tau_1) - \frac{3x(0)}{4} \\ \alpha_3 &= 6x(\tau_2) - \frac{3x(L)}{4}, \qquad \alpha_4 = x(L). \end{aligned} \tag{A.11}$$

Although $x(t) - I_{b,j}x(t)$ is no longer in the space $H_0^2(I)$, an interpolating spline $x_j(t)$ exists in the following form:

$$\begin{aligned} x_j(t) &= I_{b,j}x(t) + x_0 + y_0 + y_1 + \dots + y_{j-1} \\ x_0 &\in V_0, \quad y_i \in W_i, \qquad 0 \le i \le j-1 \end{aligned} \tag{A.12}$$

where $x_j(t)$ defined in (A.12) will still have an approximation to $x(t)$ of order $O(2^{-4j})$ [2].

### *B. DWT*

In order to efficiently compute expansion coefficients $\hat{x}_J, J \ge 0$ from function values $x(t_k^{(j)})$, $1 \le k \le L+3$, if $j = -1$ and $-1 \le k \le n_j - 2, j \ge 0$, or vice versa, we have introduced a fast DWT [2] which maps discrete sample values of a function to its wavelet interpolation expansion coefficients. To show the computational complexity, we mention the so-called point value vanishing (PVV) property of wavelet functions $\psi_{j,k}(t), j \ge 0, -1 \le k \le n_j - 2$ [2]. For $j > 0, -1 \le k \le n_j - 2$

$$\begin{aligned} &\psi_{j,k}(t_k^{(j)}) = 1, \qquad \psi_{j,k}(t_l^{(i)}) = 0, \qquad 1 \le l \le L+3 \\ &\quad \text{if } i = -1, \qquad -1 \le l \le n_i - 2, \qquad \text{if } i \ge 0. \end{aligned} \tag{B.1}$$

The following algorithm provides a recursive way to implement the transform between point value $x$ and wavelet coefficients $\hat{x}$.

(i) $\hat{x} \rightarrow x$

This direction of transform is straightforward by evaluating the expansion[1] at all collocation points $\{t_k^{(j)}\}, j \ge -1$, which takes less than or equal to $5N\log N(\mathit{flops})$ operations where the PVV property and the compactness of $\mathrm{supp}\psi_{j,k}(t)$ are used to reduce the number of evaluations and $N = 2^J L + 3$ again.

(ii) $x \rightarrow \hat{x}$

[1] Equations (2.7)–(2.13) in Section II.

We proceed to the construction of $I_J x(t)$ in three steps, i.e., the following:

Step 1) Find

$$x_{-1}(t) = I_{V_0} x^{(-1)};$$

Step 2) Find

$$x_0(x) = I_{W_0}\left(x^{(0)} - \left(I_{V_0}x\right)^{(0)}\right) = \sum_{k=-1}^{n_0-2} \hat{x}_{0,k}\psi_{0,k}(t)$$

where

$$\left(I_{V_0}x\right)^{(0)} = \left\{I_{V_0}x\left(t_k^{(0)}\right)\right\}_{k=-1;}^{n_0-2}$$

Step 3) Find

$$x_j(t) = I_{W_j}\left(x^{(j)} - (I_{j-1}x)^{(j)}\right) = \sum_{k=-1}^{n_j-2} \hat{x}_{j,k}\psi_{j,k}(t)$$

where

$$(I_{j-1}x)_k^{(j)} = I_{j-1}x\left(t_k^{(j)}\right), \quad -1 \le k \le n_j - 2.$$

So we have $I_J x(t) = x_{-1}(t) + x_0(t) + \cdots + x_{J-1}(t)$, which satisfies the required interpolation condition (2.11). Reference [2] points out that only $6N \log N(\mathit{flops})$ operations are needed.

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