

**The Optimal Stopping of "Seasonal" Observations and the  
Game of "Seasonal" Stopping.**

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## Outline

- Financial Mathematics or Real Finance/Economics ?  
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- Markov chain (MC) modulated models
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  - Stopping of Seasonal Observations
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- Optimal Stopping (OS) of Markov Chains
- State Elimination Algorithm (SEA). Solutions.
- Three Abstract Optimization Problems
- Open Problems

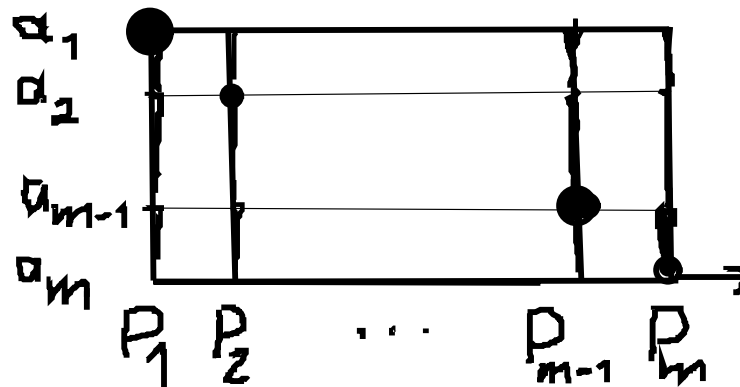
## Markov Chain Modulated Inventory Model.

joint with J. Hill

There is a firm using for production a commodity ("resource"), e.g. cotton which is consumed with a *unit intensity*. The price for this commodity is a stochastic process, modelled by a MC  $(Z_n)$  with a finite number of states  $\{1, 2, \dots, m\}$  and known transition rates  $q_{ij}$ ,  $q_i = \sum_{j \neq i} q_{ij}$ , where state  $i$  represents price  $P_i$ . A firm can buy a commodity at a current price or use the "stored" commodity. The storage cost is  $Cdt$  per unit of commodity and time interval  $dt$ , or even  $C(y(t))$ , where  $y(t)$  is a storage level at  $t$ . The goal of the firm is to minimize the average performance cost = storage cost + purchase cost [or total discounted performance cost]

storage policy  $\mathbf{a} = (a_1, a_2, \dots, a_m)$ , generates the following *buying* policy:

- do not buy if  $a_i < y$ ,
- buy with *intensity* one if  $a_i = y$ , and
- buy *instantly* an amount  $(a_i - y)$  if  $a_i > y$ .

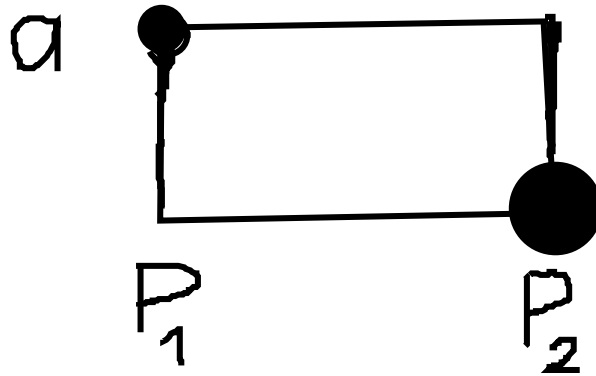


**Structure of invariant measure. Idea of solution:**

- 1) given policy  $\mathbf{a}$  find the invariant distribution of the corresponding MC;
- 2) minimize cost over all distributions.

Answer for  $m = 2$ .

If  $q_1, q_2 = q$  and  $P_1 - P_2 + C/q_1 < 0$ , then optimal  $a_1 = a$  is



$$a = -\frac{1}{q} \ln \frac{C(q+q_1)}{q_1(P_2q - P_1q + C)}.$$

If  $P_1 - P_2 + C/q_1 \geq 0$ , then optimal  $a_1 = 0$ .

Even for  $m = 3$  the answer is very complicated and this case is not fully studied.

## Optimal Stopping of "Seasonal" observations

The following problem of optimal stopping (OS) of MC was dubbed previously by an author as OS of "Seasonal" Observations.

A player (decision maker) observes a MC  $(Z_n)$  with two components, where the first one is an "underlying" finite MC  $(U_n)$  with  $m$  states and given transition matrix  $Q$ , and the other component is one of  $m$  independent sequences of i.i.d. r.v.  $(Y_n(k))$  with known distributions  $F = \{f(k, \cdot)\}$ .

A simple example of such a situation is a case in which there are two dice with four and six equally likely sides and the determination as to which of them is tossed at a given moment is specified by a position of a Markov chain with two states and given stochastic matrix  $P$ .

The goal of a player is to maximize the discounted expected reward over all possible stopping times.

## The Game of Seasonal stopping

is a game where in a similar setting there are  $m$  players, and  $k$ -th player flips a die with known distribution  $F(k, \cdot)$  when the underlying MC  $U_n$  is in her "favorable" state  $k$ . After the observation  $i$  she can stop and obtain a reward  $g(k, i)$ . Then the game is over and the other players obtain zero. If she continues then MC moves into a new state and so on. The goal of each player is to maximize the expected reward. A policy of  $k$ -th player is a stopping set  $S(k)$ .

*More general setting:* there are  $N$  players,  $N \leq m$  and a state space  $B$  is partitioned into  $N + 1$  sets  $B(k), B(k) \neq \emptyset$  for  $k = 1, 2, \dots, N$ . A player  $k$  can stop the game only when the underlying MC  $U_n$  is in her "favorable" subset  $B(k)$ . In a state  $s \in B(k)$  she flips a die with known distribution  $F(s, \cdot)$ .

*Even more general setting*:...if at state  $x$  all players continue then the state of an underlying MC is moved according to stochastic matrix  $Q$ , if a nonempty set of players  $J$  stops then with probability  $s(x|J)$  game is over and  $k$ -th player obtains reward  $K_k(x|J)$ , with complementary probability the game proceeds as in continue mode...

*Nash equilibrium* (NE) = a set of strategies such that  $i$ -th player can not benefit if she deviates from her strategy and all other players do not.  $\mathbf{E}$  = set of all NE

**Two goals:**

(Easy) To check whether *given* strategy  $\pi$  is an NE, i.e.  $\pi \in \mathbf{E}$ .

(Hard) To list all NE efficiently.

Let for simplicity  $B = N = \{1, 2, \dots, m\}$ .

**To stop or not to stop, that is the question !**

OS: probability models, the pricing of American options, building block...

There are two approaches - " martingale theory of OS " and "Markovian approach"

classical monographs Chow, Robbins and Sigmund (1971)

A. Shiriyayev (1969, 1978)

T. Ferguson (website)

G. Peskir, A. Shiriyayev (2006)

**T. Ferguson: "Most problems of optimal stopping without some form of Markovian structure are essentially untractable..."**

**Optimal Stopping Problem of MC**  $M = (X, A, P, c, g, \beta)$ ,  
 $P = \{p(x, y)\}$ ,  $(Z_n)$ ,  $P_x, E_x, A = \{\text{cont.}, \text{stop}\}$ ,  
 $Pf(x) = \sum_y p(x, y)f(y)$ ,  $X$  finite (countable),  $c(x)$  one step cost  
function,  $g(x)$  terminal reward function,  $\beta$  discount factor,  $\beta \leq 1$ ,  
 $v(x) = \sup_{\tau \geq 0} E_x[\sum_{i=0}^{\tau-1} \beta^i c(Z_i) + \beta^\tau g(Z_\tau)]$  value function

**Remark !** absorbing state  $e$ ,  $p(x, y) \rightarrow \beta p(x, y)$ ,  $p(x, e) = 1 - \beta$ ,  
 $\beta \rightarrow \beta(x) = P_x(Z_1 \neq e)$  probability of "survival"  
 $S = \{x : g(x) = v(x)\}$  optimal stopping set

**Theorem 1.** (Shiryaev 1969). (a) *The value function  $v(x)$  is the minimal solution of Bellman equation ...*

$$v = \max(g, c + Pv)$$

(e) *if state space  $X$  is finite then set  $S$  is not empty and  $\tau_0 = \min\{n \geq 0 : Z_n \in S\}$  is an optimal stopping time.*

## Basic methods of solving OS of MC $c \equiv 0$

- the direct solution of the Bellman equation
- the value iteration method : one considers the sequence of functions

$$v_n(x) = \sup_{0 \leq \tau \leq n} E_x \dots, \quad v_{n+1}(x) = \max(g(x), P v_n(x)), \\ v_0(x) = g(x). \text{ Then } v_0(x) \leq v_1(x) \leq \dots; \quad v_n(x) \text{ converges to } v(x).$$

- the linear programming approach ( $|X| < \infty$ ),  $\min \sum_{y \in X} v(y)$ ,  
 $v(x) \geq \sum_y p(x, y) v(y), v(x) \geq g(x), x \in X$ .

- Davis and Karatzas (1994), interesting interpretation of the Doob-Meyer decomposition of the Snell's envelope

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- The *State Elimination Algorithm* (SEA) Sonin (1995, 1999, 2005)  
E. Presman modification (2009)

OS = Bellman equation

$$v(x) = \max(g(x), P_1 v(x)); M_1 = (X_1, P \equiv P_1, g), v = v_1$$

**The State Elimination algorithm (SEA):** three facts:

1. It may be *difficult* to find the states where it is optimal *to stop*, but it is *easy* to find a state (states) where it is optimal *not to stop*:

stop if  $g(x) \geq P_1 v(x)$  but  $v$  is unknown, so...

do not stop if  $g(z) < P_1 g(z) \leq P_1 v(z)$  or...

2. Then "eliminate"  $z$ , (set  $D$ ) and recalculate  $P_1 \rightarrow P_2$  and  $c_1 \rightarrow c_2, g$ . *Elimination theorem:*  $S_1 = S_2, v_1 = v_2$ . After that repeat the first step and so on.

3. Finally, there is a well known

**Proposition 1.** Let  $M = (X, P, g)$  be an optimal stopping problem, and  $g(x) \geq P g(x)$  for all  $x \in X$ . Then  $X$  is the optimal stopping set in the problem  $M$ , and  $v(x) = g(x)$  for all  $x \in X$ .

What means to eliminate state  $z$ , (set  $D$ ) and recalculate probabilities ? Embedded Markov chain (Kolmogorov, Doeblin)

$M_1 = (X_1, P_1)$ ,  $D \subset X_1$ ,  $X_2 = X_1 \setminus D$ ,  $(Z_n)$  MC

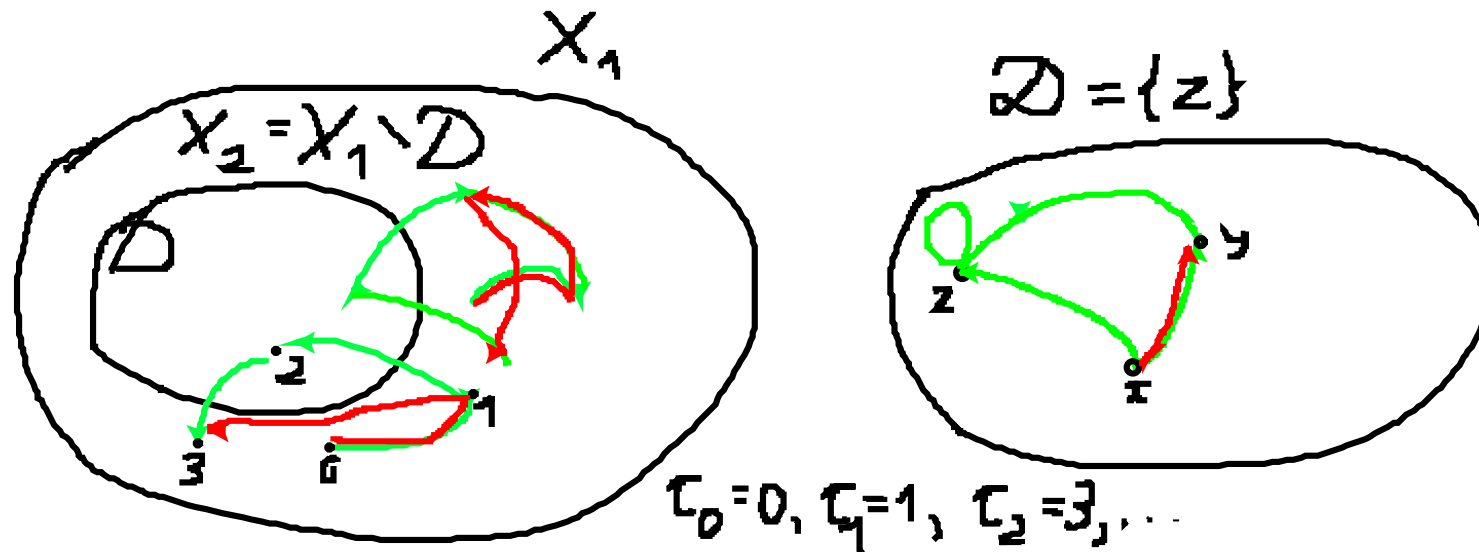
$\tau_0, \tau_1, \dots, \tau_n, \dots$ , the moments of zero, first, and so on, visits of  $(Z_n)$  to the set  $X_2$ .  $Y_n = Z_{\tau_n}, n = 0, 1, 2, \dots$

**Lemma 1.** (Kolmogorov, Doeblin) (a) *The random sequence  $(Y_n)$  is a Markov chain in a model  $M_2 = (X_2, P_2)$ , where  $P_2 \dots$*

If  $D = \{z\}$  then

$$p_2(x, \cdot) = p_1(x, \cdot) + p_1(x, z)n_1(z)p_1(z, \cdot) ,$$

where  $n_1(z) = 1 + p_1(z, z) + (p_1(z, z))^2 + \dots = 1/(1 - p_1(z, z))$ .



If  $D = \{z\}$  then

$$p_2(x, y) = p_1(x, y) + p_1(x, z)n_1(z)p_1(z, y),$$

where  $n_1(z) = 1 + p_1(z, z) + (p_1(z, z))^2 + \dots = 1/(1 - p_1(z, z))$ .

For general set  $D$  the transition matrix  $P_2 = \{p_2(x, y)\}$  is given by the formula

$$p_2(x, y) = p_1(x, y) + \sum_{z \in D} p_1(x, z)u_1(z, y), \quad (x, y \in X_2).$$

$$P_1 = \begin{bmatrix} Q_1 & T_1 \\ R_1 & P_{01} \end{bmatrix}, \quad P_2 = P_{01} + R_1 U_1 = P_{01} + R_1 N_1 T_1,$$

$N_1$  is a (transient) *fundamental matrix*, i.e.  $N_1 = (I - Q_1)^{-1}$ ,

$$N = I + Q + Q^2 + \dots = (I - Q)^{-1}, U = NT.$$

*Full size* formula for new stochastic matrix (E. Presman modification)

$$P_D = (I + P I_D N_D) P I_S = N_D P I_S,$$

where  $N_D = (I - P I_D)^{-1}$  and  $I_B$  is a diagonal matrix with  $d_i = 1$  if  $i \in B$  and  $0$  otherwise.

$$P_D = \begin{bmatrix} O & N_1 T_1 \\ O & P_2 \end{bmatrix}, \quad P_2 = P_{01} + R_1 U_1 = P_{01} + R_1 N_1 T_1,$$

$N_1$  is a (transient) *fundamental matrix*, i.e.  $N_1 = (I - Q_1)^{-1}$ ,

or  $P_D = (I + P I_D N_D) P I_S = N_D P I_S$ ,

where  $N_D = (I - P I_D)^{-1}$  and  $I$  is the identity matrix of full size.

After the elimination... the stopping sets and the value functions in the remaining points will coincide.

*Elimination theorem:* (Sonin, 1995, 1999)

**SE algorithm**

## SE algorithm

$$g(x) - (c_1(x) + P_1g(x)) = g - F_1g$$



$$g(x) - F_1g(x) \geq 0 \text{ for all } x$$

$$\text{there is } z : g(z) - F_1g(z) < 0$$



$$X_1 = S$$

$$M_1 \longrightarrow M_2 : g(x) - F_2g(x)$$



... and so on/

On the 2nd stage we recursively calculate  $v(x)$ . With full size formulas we do not need the 2nd stage at all:  $v_n \nearrow v$ , where  $v_n$  is "value function" obtained on  $n$ -th step of elimination, i.e. (with  $c = 0$ ),  $v_n = E_x g(Z_{\tau_n})$ ,  $\tau_n$  is the moment of the first visit to set  $S_n = X \setminus D_n$ .

## Finite State Space

If  $|X| = k < \infty$  then SEA solves OS Problem in  $\leq 2k$  steps

## Countable or General State Space

- Random walk on a line, but not in a plane
- Optimal Stopping of "Seasonal" observations  
Sonin (2001), Presman and Sonin (2008), submitted
- Game: Optimal Stopping of "Seasonal" observations
- *A generalized Gittins index for a Markov chain and its recursive calculation.* Sonin in Stat. & Prob. Let., Sept. 2008.
- *The OS of MC and Recursive Solution of Poisson and Bellman Equations.* Sonin in Shiryaev Festschrift, Springer, 2006.
- Next talk: C. Steinberg , I. Sonin: Continue, Stop, Restart  
Probability Model

## Solution to Stopping of "Seasonal" observations

Formally,  $M = (\mathbf{X}, \mathbf{P}, g)$ ,  $\mathbf{X} = (B \times X) \cup e$ ,  $B = \{1, \dots, m\}$ ,  $X = R$  or  $Z = \{1, 2, \dots\}$ ,  $\mathbf{P} = Q \times F$ ,  $Q = \{q(s, k)\}$ ,  $s, k \in B \cup e$ ,  $x = (s, i)$ ,  $c(x) = c(s)$ ,  $c(x) \equiv 0$ ,  $F = \{f(k, \cdot)\}$ ,  $g(x) = g(s, i)$ .  $(Z_n) = ((U_n), (Y_n(U_n)))$ .

**Theorem 1.** (a) *There are threshold values  $c_*(1), \dots, c_*(m)$ ,  $c_*(i) \leq \infty$ , such that the optimal stopping set  $S_* = \{S_*(k)\}$ ,  $S_*(k) = \{(k, i) : g(k, i) \geq c_*(k)\}$ ,  $k = 1, \dots, m$ ;*

(b) *Only a finite number of steps  $n = n_*$  is necessary to obtain an optimal stopping set using SEA;*

(c) *The value function  $v(k, i) = g(k, i)$  if  $i \in S_*(k)$  and  $v(k, i) = w_*(k)$  if  $i \notin S(k)$ , where  $w_*(k) = \mathbf{P}_* g(k, \cdot)$ ,  $\mathbf{P}_* = \mathbf{P}_{n_*}$ ,  $n = n_*$ ,  $\mathbf{P}_* = Q_* \times F_*$ .*

Notation:  $F_*(k, \cdot) = \{f(k, \cdot)/f(k, S_*(k))\}$ . Given set  $A = \{A(k)\}$ ,  $f(k, S(k)) = \sum_{i \in S(k)} f(k, i)$ , i.e. just a normalized initial distribution,  $F_A$  a diagonal  $m \times m$  matrix with diagonal elements  $f_A(k) = \sum_{i \in A(k)} f(k, i)$  and  $N_A = (I - QF_A)^{-1}$ , a fundamental matrix for  $QF_A$ . Then  $Q_* = N_{D_*} QF_{S_*}$ .

**Example 1, N=1.**

$B = \{1, 2\}$ , die 1 = {1, 2, 3, 4}, die 2 = {1, 2, ..., 5, 6},  $\beta = .9$ ,

$$P = \begin{array}{|c|c|} \hline 3/4 & 1/4 \\ \hline 1/3 & 2/3 \\ \hline \end{array}, \quad P(\beta) = \begin{array}{|c|c|c|} \hline 27/40 & 9/40 & 1/10 \\ \hline 9/30 & 18/30 & 1/0 \\ \hline 0 & 0 & 1 \\ \hline \end{array},$$

$$S = \{4; 4, 5, 6\}, \quad w_*(1) = 3 \frac{69}{205}, \quad w_*(2) = 3 \frac{135}{205}.$$

## Nash Equilibrium Points (NE). Characteristics. Some examples

Let us denote  $\mathbf{E}$  the set of all Nash equilibrium points for this problem. Without loss of generality a strategy of a  $k$ -th player is defined by her stopping set  $S(k)$  and this stopping set is defined by a threshold value  $c(k)$ , i.e.  $S(k) = \{i : g(k, i) \geq c(k)\}$ . Thus any tuple of strategies  $\pi$  can be denoted also as  $\pi = \{c(j), j \in N\}$ . Given  $\pi = \{S(j), j \in N\}$  and initial state of the game  $x = \{k, i\}$  the payoff of  $k$ -th player is equal to  $w^\pi(k) = E_x g(k, Z_\tau)$ , where  $\tau$  is the moment of a first visit to stopping set  $S = \cup S(j) \cup e$ . Because of a special structure of MC this value depends only on the first coordinate  $k$ . Let us denote  $m^\pi(k) = \max \{g(k, i) : i \notin S(k)\}$ .

**Theorem 2.** *There are "upper" and "lower" NE, which may coincide.*

**Example 2, N=2.**

$B = \{1, 2\}$ , die 1 = {1, 2, 3, 4}, die 2 = {1, 2, ..., 5, 6},  $\beta = 1$ ,

$$P = \begin{array}{|c|c|} \hline 3/4 & 1/4 \\ \hline 1/3 & 2/3 \\ \hline \end{array}$$

There is a unique  $\pi \in \mathbf{E}$ ,  $\pi = \{3; 3\}$ ,  $w_*(1) = 2\frac{9}{34}$ ,  $w_*(2) = 2\frac{31}{34}$ ;

**Example 3, N=2.**  $B = \{1, 2\}$ , die 1 = {1, 2, 3}, die 2 =

$$\{1, 2, 3, 4\}, \beta = 1, \quad P = \begin{array}{|c|c|} \hline 3/4 & 1/4 \\ \hline 1/3 & 2/3 \\ \hline \end{array}$$

There are two  $\pi \in \mathbf{E}$ ,  $\pi_1 = \{2; 2\}$ ,  $w_*(1) = 1\frac{66}{89}$ ,  $w_*(2) = 1\frac{82}{89}$ ;

$\pi_2 = \{2; 3\}$ ,  $w_*(1) = 1\frac{21}{34}$ ,  $w_*(2) = 2\frac{10}{34}$ .

## Three abstract optimization problems

Suppose there is an index set  $U$ , and  $A = a_u$  and  $B = b_u$  be two sets indexed by the elements of  $U$ . Suppose that an assumption  $U$  holds,

$$a_u = a < \infty, \quad 0 < b = b_u = 1 < \infty. \quad (U)$$

**Problem 1. Restart problem.** (Katehakis, Veinott index)

Find solution(s) of the equation

$$h = H(h), \text{ where } H(h) = \sup_{u \in U} [a_u + (1 - b_u)h]. \quad (*)$$

There are two equivalent interpretations of this problem.

There is a set of "buttons"  $u \in U$ . A DM can select one of them and push. She obtains a reward  $a_u$  and according to the first interpretation with probability  $b_u$  the game is *terminated*, and with complimentary probability  $1 - b_u$  she can select any button again. Her goal is to maximize the total (undiscounted) reward.

According to the second interpretation the game is continued sequentially and  $1 - b_u$  is not the probability but a *discount factor* applied to the future rewards. It can be easily proved that in both cases her value satisfies the equation above.

Our second optimization problem is

**Problem 2. Ratio (cycle) problem** Find  $\alpha$

$$\alpha = \sup_{u \in U} \frac{a_u}{b_u} \quad (\text{Gittins index}) \quad (**)$$

The interpretation of this problem is straightforward: a DM wants to maximize the ratio of the one step reward per chance of termination.

**Problem 3. A parametric family of Retirement problems**

Find  $w$ , (Whittle index) defined as follows: given parameter  $k$ ,  $-\infty < k < \infty$ , let

$$v(k) = \max(k, H(k)), \quad w = \inf\{k : v(k) = k\}.$$

$$h = \sup_{u \in U} [a_u + (1 - b_u)h]. \quad (*)$$

$$\alpha = \sup_{u \in U} \frac{a_u}{b_u} \quad (**)$$

### **Theorem 1**

- a)  $\alpha < \infty$  and is a (unique) solution of equation (\*);
- b)  $\alpha = h = w$ ;
- c) the optimal index  $u$ , if exists, or an optimizing sequence  $u_n$  coincide for all three problems.

## Open Problems.

- analog of SEA in continuous time/space
- more examples of Seasonal Stopping and Game of SOS for different distributions  $F(k, \cdot)$ , both discrete and continuous
- multiple restart problems
- multidimensional equivalent abstract optimization problems
- explanation of world financial crisis

*Thank you for your attention !*