The Optimal Stopping of Markov Chain and its Application to other Probability Problems

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Outline

- (Pure) Probability Theory
- Many-armed Bandit problems (with dependent arms)
- Economic Theory. Games
- Applied Probability: secretary problem, optimal stopping (OS) problems, sufficiency of Markov, stationary strategies in MDP, replacement models, inventory models, *State Elimination Algorithm, Generalized Gittins index*
    - Optimal Stopping of Markov chains
    - State Reduction Approach
    - Generalized Gittins index. Four indices
    - Abstract Optimization Problems
    - Open Problems. Plans

Markov model $M = (X, P), \;\; P = \{p(x, y)\}, \;\; (Z_n), \;\; U_0 = \{(Z_n)\}$

family of all finite Markov chains

The behavior of U_0 is a classical result of Probability Theory derived in the 30's by A. Kolmogorov and W. Doeblin.

If P is replaced by (P_n) then ... U... family of *nonhomogeneous* MCs.

Is it possible to say something about the behavior of the family U if there are *no assumptions* about sequence (P_n) ?

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Is it possible to say something about the behavior of the family U if there are *no assumptions* about sequence (P_n) ?

The surprising answer to this question is Yes. Such behavior is described by a theorem which we call a Decomposition-Separation (DS) Theorem. Simple deterministic interpretation plus martingales.

Net Present value (NPV) and **Internal rate of return (IRR)**

IRR = effective interest rate = yield to maturity

Income stream = *cash flow* = project $a = (a_0, a_1, ..., a_T)$.
If $a_i < 0$ (is negative) it means that an investor pays money, if its positive she receives money. For this cash flow we can calculate NPV for a particular interest rate r. If we set NPV equal to zero, the roots of this equation will give the IRsR

$$NPV(r) = a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \dots \frac{a_T}{(1+r)^T} = 0.$$

Investment polynomial $\sum_{0 \le i \le T} a_i z^i = 0,\ z = 1/(1+r)$. Simplest case: you invest a_0 and after moment T you receive a_T, equation $|a_0|(1+r)^T = a_T$. In this case there is exactly one root; in general there may be many roots. The interpretation of multiple roots of IRR is given in my paper *Economic Theory*, 5, (1995). Presman, Sonin (200X)

To stop or not to stop, that is the question !

OS: many probability models, and now the pricing of American options

there are two approaches - " martingale theory of OS " and "Markovian approach"

classical monographs Chow, Robbins and Sigmund (1971)

A. Shiryayev (1969, 1978)

T. Ferguson (website)

G. Peskir, A. Shiryayev (2006)

T. Ferguson: "Most problems of optimal stopping without some form of Markovian structure are essentially untractable...".

Optimal Stopping Problem of Markov Chain (MC)

$M = (X, P, c, g, \beta)$, $P = \{p(x,y)\}$, (Z_n), P_x , E_x, $Pf(x) = \sum_y p(x,y) f(y)$, X finite (countable), $c(x)$ *one step cost* function, $g(x)$ *terminal reward* function, β discount factor, $\beta \leq 1$,

$v(x) = \sup_{\tau \geq 0} E_x[\sum_{i=0}^{\tau-1} \beta^i c(Z_i) + \beta^\tau g(Z_\tau)]$ *value function*

Remark: absorbing state e, $p(x,y) \longrightarrow \beta p(x,y)$, $p(x,e) = 1 - \beta$, $\beta \longrightarrow \beta(x) = P_x(Z_1 \neq e)$ probability of "survival"

Bellman (optimality) equation $v = \max(g, c + Pv) = \max(g, Fv)$

$S = \{x : g(x) = v(x)\}$ optimal stopping set

Theorem 1. (Shiryaev 1969). (a) *The value function* $v(x)$ *is the minimal solution of Bellman equation ...*

(e) *if state space* X *is finite then set* S *is not empty and* $\tau_0 = \min\{n \geq 0 : Z_n \in S\}$ *is an optimal stopping time.*

Basic methods of solving OS of MC

- the direct solution of the Bellman equation
- the value iteration method : one considers the sequence of functions $v_n(x) = \sup_{0 \le \tau \le n} E_x ...,\ v_{n+1}(x) = \max(g(x), Fv_n(x)),$ $v_0(x) = g(x)$. Then $v_0(x) \le v_1(x) \le ...; \quad v_n(x)$ converges to $v(x)$.
- the linear programming approach $(\ |X| < \infty),\ \ \min \sum_{y \in X} v(y),$ $v(x) \ge \sum_y p(x,y)v(y), v(x) \ge g(x), x \in X.$
- Davis and Karatzas (1994), interesting interpretation of the Doob-Meyer decomposition of the Snell‘s envelope
- The State Elimination Algorithm (EA) Sonin (1995, 1999, 2005)

Goal: a unified recursive approach to

Problem 1 To find a minimal solution v of a *Bellman equation*

$$v = \max(g, c + Pv) = \max(g, Fv)$$

Free boundary problem

Problem 2 To find the solution f of the discrete *Poisson equation*

$$f = c + Pf$$

Problem 3 To calculate *Gittins index (GI)* $\gamma(x)$ and the Generalized Gittins index $\alpha(x)$,

$$\gamma(x) = \sup_{\tau > 0} \frac{E_x \sum_{n=o}^{\tau-1} \beta^n c(Z_n)}{E_x \sum_{n=o}^{\tau-1} \beta^n},$$

i.e. $\gamma(x)$ is the maximum of a *expected discounted reward* per *expected discounted unit of time*, $0 < \beta < 1$, τ is a stopping time.

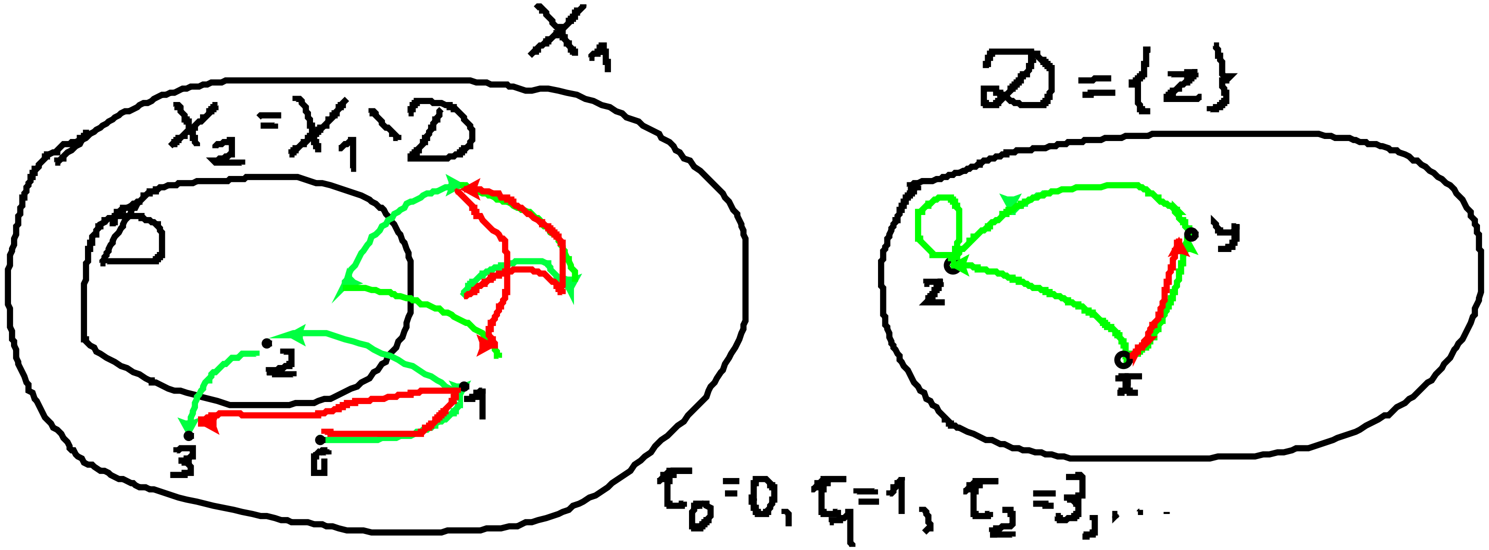
State Reduction (SR) approach

$M_1 = (X_1,\, P_1),\;\; D \subset X_1,\, X_2 = X_1 \setminus D,\, (Z_n), n = 0, 1, 2, \ldots$

$\tau_0, \tau_1, \ldots, \tau_n, \ldots,$ the moments of zero, first, and so on, visits of (Z_n) to the set X_2 : $\tau_0 = 0,\, \ldots\; \tau_{n+1} = \min\{k > \tau_n, Z_k \in X_2\}$,

$(Y_n),\quad Y_n = Z_{\tau_n},\quad n = 0, 1, 2, \ldots$.

$U_1 = \{u_1(x, y)\},\, u_1(x, y) = P_x(Z_{\tau_1} = y), x \in D,\; y \in X_2.$



Lemma 1. (Kolmogorov, Doeblin)

(a) (Y_n) *is a Markov chain in a model* $M_2 = (X_2, P_2)$, *where*

(b) *the transition matrix* $P_2 = \{p_2(x,y)\}$ *is given by the formula*

$$p_2(x,y) = p_1(x,y) + \sum_{z \in D} p_1(x,z) u_1(z,y), \quad (x, y \in X_2).$$

$$P_1 = \begin{bmatrix} Q_1 & T_1 \\ R_1 & P_1' \end{bmatrix}, \quad P_2 = P_1' + R_1 U_1 = P_1' + R_1 N_1 T_1, \tag{1}$$

N_1 is a (transient) *fundamental matrix*, i.e. $N_1 = (I - Q_1)^{-1}$,

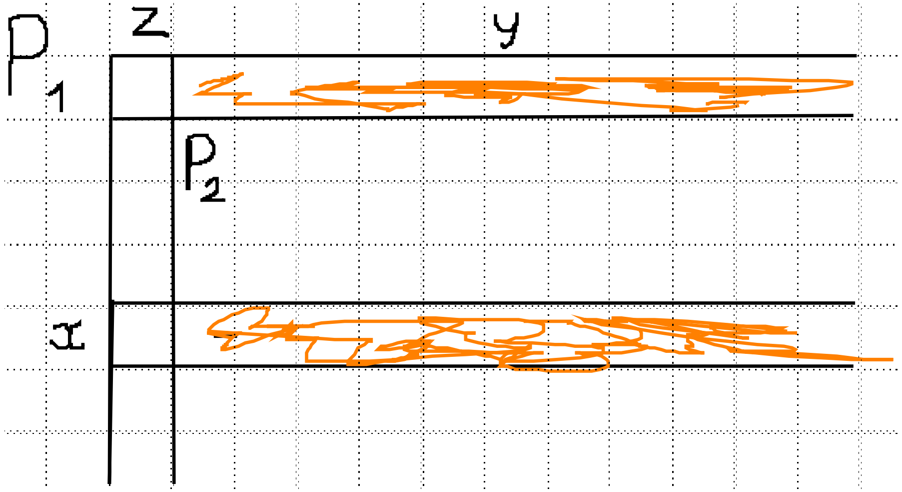
$N = I + Q + Q^2 + \ldots = (I - Q)^{-1}, U_G = \{u(x,s) = P_x(Z_{\tau_G} = s\},$

$U = NT$

If $D = \{z\}$ then

$$p_2(x,\cdot) = p_1(x,\cdot) + p_1(x,z)n_1(z)p_1(z,\cdot)\ , \tag{2}$$

where $n_1(z) = 1 + p_1(z,z) + (p_1(z,z))^2 + \ldots = 1/(1 - p_1(z,z))$.



Lemma 2. *Let* M_1 *and* M_2 *be two adjacent models with state spaces* X_1 *and* $X_2 = X_1 \setminus D$*, where* $D \subseteq X_1, G \subset X_2, N_1, N_2, U_1, U_2$*. Then* $u_1(x,y) = u_2(x,y), \quad y \in G$, $n_1(x,v) = n_2(x,v)$.

$$N_1 = N_2, \qquad\qquad U_1 = U_2$$

$M_1 = (X_1,\, P_1, c_1, g), \quad c_1(x)$ one step cost function transformation

If $D = \{z\}$ then (T.J. Sheskin (1999))

$$c_2(x) = c_1(x) + p_1(x,z)n_1(z)c_1(z), \ \ x \in X_2.$$

For the general set D, in matrix form

$$c_2 = c_{1,X_2} + R_1 N_1 c_{1,D} \tag{3}$$

OS $=$ Bellman equation $v(x) = \max(g(x), c(x) + Pv(x))$

The State Elimination algorithm (SEA): three facts:

1. It may be *difficult* to find the states where it is optimal *to stop*, but it is *easy* to find a state (states) where it is optimal *not to stop.*

stop if $g(x) \geq c(x) + Pv(x)$ but v is unknown, so...

do not stop if $g(z) < c(z) + Pg(z) \leq c(z) + Pv(z))$ or...

2. Then eliminate z, (set D) and recalculate $P_1 \longrightarrow P_2$ and $c_1 \longrightarrow c_2$, g. Lemma 2 implies *Elimination theorem:* $S_1 = S_2, v_1 = v_2$ for $x \in X_2$. After that repeat the first step and so on.

3. Finally, there is a well known

Proposition 1. *Let* $M = (X, P, g)$ *be an optimal stopping problem, and* $g(x) \geq c(x) + Pg(x)$ *for* all $x \in X$. *Then* X *is the optimal stopping set in the problem* M, *and* $v(x) = g(x)$ *for all* $x \in X$.

After the elimination... the stopping sets and the value functions in the remaining points will coincide

SE algorithm

$$\begin{tikzcd}
 & g(x)-(c_1(x)+P_1g(x)) = g - F_1g \arrow[dl] \arrow[dr] & & \\
g(x)-F_1g(x)\geq 0 \ \text{for } \textit{all } x \arrow[d, Rightarrow] & & \text{there is } z : g(z)-F_1g(z)<0 \arrow[d, Rightarrow] & \\
X_1 = S & & M_1 \longrightarrow M_2 : g(x)-F_2g(x) \arrow[dl] \arrow[dr] & \\
 & {} & & {} \\
 & & \dots &
\end{tikzcd}$$

On the 2nd stage we recursively calculate $v(x)$

Sonin (1995); Sonin, I.M.: The Elimination Algorithm for the Problem of Optimal Stopping. *Math. Meth. of Oper. Res.*, **49**, no. 1, (1999)

Sonin, I.M.: The State Reduction and related algorithms and their applications to the study of Markov chains, graph theory and the Optimal Stopping problem. *Advances in Mathematics* **145**, no. 2, (1999)

SR Algorithms 1985, GTH/S

Grassman, Taksar, Heyman; Sheskin

$$M_1 = (X_1, P_1), |X_1| = k,$$

$$P_1 \longrightarrow P_2 \longrightarrow ... \longrightarrow P_k = \{1\}$$

$$\pi_1 \longleftarrow \pi_2 \longleftarrow ... \longleftarrow \pi_k = \{1\}$$

GTH Algorithm to calculate the invariant distribution of MC
GTH/S

1. T. J. Sheskin *Oper. Res.* **33** (1985).

2. W. K. Grassmann, M. I. Taksar, and D. P. Heyman, *Oper. Res.* **33** (1985).

other SR algorithms: FUND Heyman (1995), REFUND Sonin and Thornton (1999, 2001), Kohlas (1986),...

Countable State Space

- Random walk on a line
- Optimal Stopping of ”Seasonal observations”

There are m different ”dice” but which die to observe at moment n is specified by a position of an underlying Markov chain (U_n) independent of observations on dice

Sonin (2001), Presman and Sonin (2008), submitted

- Game: Optimal Stopping of ”Seasonal observations”

  There are m different ”dice” and $N, N \leq m$ players, player i can stop the game when her dice is observed

Recursive Solution of Poisson equation

$$f = c + Pf$$

Lemma 3. *Let* M_1 *and* M_2 *be two adjacent models with state spaces* X_1 *and* $X_2 = X_1 \setminus D$*, where* $D \subseteq X_1$*,* P_i *and* $F_i, i = 1, 2$ *be the corresponding averaging and reward operators, where functions* c_1 *and* c_2 *are related by* (3)*, matrices* R_1, T_1 *are as in* (1) *and matrix* N_1 *is a fundamental matrix for* Q_1*. Let* f *be a function defined on* X_1*. Then*

$$f_{X_2} - F_2 f_{X_2} = (f - F_1 f)_{X_2} + R_1 N_1 (f - F_1 f)_D.$$

$$f_D = N_1[(f - P_1 f)_D + T_1 f_{X_2}]$$

Theorem 2. *Let* M_1 *and* M_2 *be two adjacent models with state spaces* X_1 and $X_2 = X_1 \setminus z$, *where* P_1 *and* P_2, *functions* c_1 *and* c_2 *are related by* (2) and (3).

Then a function f *satisfies an equation*

$$f = c_1 + P_1 f$$

if and only if its restriction to X_2 *satisfies an equation*

$$f = c_2 + P_2 f$$

and $$f(z) = n_1(z)[\textstyle\sum_{y \in X_2} p_1(z,y) f(y) + c_1(z)] \qquad (4)$$

A new recursive algorithm:

$|X| = k, \quad \sum c_1(y)\pi_1(y) = 0 \Rightarrow \sum c_2\pi_2 = 0;$

$$P_1, c_1 \longrightarrow P_2, c_2 \longrightarrow \ldots \longrightarrow P_k = \{1\}, c_k = 0$$

$$f_1 \quad \longleftarrow f_2 \quad \longleftarrow \ldots \qquad \ldots \longleftarrow \quad f_k = 0$$

Example 1. $X_1 = \{1, 2, 3\}$, P_1, $\pi(1) = \frac{12}{35}$, $\pi(2) = \frac{14}{35}$, $\pi(2) = \frac{9}{35}, (c_1, \pi) = 0, c_1(1) = 3, c_1(2) = 2,$ and $c_1(3) = -\frac{64}{9}$.

$$P_1 = \begin{array}{|c|c|c|} \hline 1/3 & 1/3 & 1/3 \\ \hline 1/4 & 1/2 & 1/4 \\ \hline 1/2 & 1/3 & 1/6 \\ \hline \end{array}, \; P_2 = \begin{array}{|c|c|} \hline 5/8 & 3/8 \\ \hline 7/12 & 5/12 \\ \hline \end{array}, P_3 = \{1\}$$

By eliminating state 1 we obtain transition matrix P_2, and function $c_2, c_2(2) = 2 + \frac{1}{4}\frac{3}{2}3 = \frac{25}{8}$, and $c_2(3) = -\frac{64}{9} + \frac{1}{2}\frac{3}{2}3 = -\frac{175}{36}$. Now eliminate state 2. Then $P_3 = \{1\}, c_3(3) = 0$. Take e.g. $f(3) = 0$. Then $f(2) = \frac{8}{3}\frac{25}{8} = \frac{25}{3}$. Then for $n = 1$, we obtain $f(1) = \frac{3}{2}[\frac{1}{3}\frac{25}{3} + 3] = \frac{26}{3}$. If $(f, \pi) = 0$, we can set $f(1) = \frac{26}{3} + k, f(2) = \frac{25}{3} + k, f(3) = k$ and to find $k = -\frac{662}{105}$. Then $f(1) = \frac{248}{105}, f(2) = \frac{213}{105}$, and $f(3) = -\frac{662}{105}$.

Recursive Solution of Bellman equation $v = \max(g, c + Pv)$.

Theorem 3. (For $D = \{z\}$). *Let* M_1 *and* M_2 *be two adjacent models with state spaces* X_1 and $X_2 = X_1 \setminus z$, *where* $g(z) \leq c_1(z) + P_1 g(z)$, P_1 *and* P_2, *functions* c_1 *and* c_2 *are related by* (2) *and* (3).

Function f *is a (minimal) solution of Bellman equation*

$$f = \max(g, c_1 + P_1 f) \textit{ on } X_1$$

if and only if $f = c_1 + P_1 f$ *on* D, *its restriction to*X_2 *is a (minimal) solution of Bellman equation*

$$f = \max(g, c_2 + P_2 f) \textit{ on } X_2,$$

and the restrictions f_{X_2} *and* f_D *are related by formula*

$$f(z) = n_1(z)[\sum_{y \in X_2} p_1(z, y) f(y) + c_1(z)]$$

Example 2. Let $X_1 = \{1, 2, 3\}$, transition probabilities are given by a matrix P_0, $P_0 = P_1$ from an Example 1, cost function $c(x)$ is: $c_1(1) = 1, c_1(2) = -.5, c_1(3) = .5$, the terminal reward function $g(x)$ is: $g(1) = -1, g(2) = 2, g(3) = 3.5$, and discount factor $\beta = .9$.

$$P_0 = \begin{array}{|l|l|l|}\hline .33 & .33 & .34 \\ \hline .25 & .5 & .25 \\ \hline .5 & .33 & .17 \\ \hline \end{array}, \qquad \mathrm{P}_1 = \begin{array}{|l|l|l|l|}\hline .3 & .3 & .3 & .1 \\ \hline .225 & .45 & .225 & .1 \\ \hline .45 & .3 & .15 & .1 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}.$$

The first step: consider $g(x) - T_1 g(x) \equiv g(x) - (c_1(x) + P_1 g(x))$ and obtain $g(1) - T_1 g(1) = -3.35 < 0$. Therefore state 1 can be eliminated. Then P_2, $c_2(2) = -.18$, and $c_2(3) = 1.14$. At the second step $g(2) - T_2 g(2) = -.04 < 0$ and state 2 can be eliminated.

$$P_2 = \begin{array}{|l|l|l|}\hline .54 & .32 & .13 \\ \hline .50 & .34 & .16 \\ \hline 0 & 0 & 1 \\ \hline \end{array}, \qquad \mathrm{P}_3 = \begin{array}{|l|l|}\hline .69 & .31 \\ \hline 0 & 1 \\ \hline \end{array}$$

P_3, $c_3(3) = .95$.. Then $g(3) - T_3 g(3) = .13 > 0$ and therefore $S = \{3, x_*\}$, and $v(3) = g(3) = 3.5$. Applying formula (4) for $n = 2$, we obtain $v(2) = \frac{1}{.46}[.32(3.5) - .18] = 2.043$. Applying formula (4) again for $n = 1$, we obtain $v(1) = \frac{1}{.7}[.3(2.043) + .3(3.5) + 1] = 3.804$.

Gittins Index (GI) $\gamma(x)$ and Generalized GI $\alpha(x)$

$$\gamma(x) = \sup_{\tau>0} \frac{E_x \sum_{n=0}^{\tau-1} \beta^n c(Z_n)}{E_x \sum_{n=0}^{\tau-1} \beta^n}$$

Multi-armed bandit (MAB) problems with *independent* arms
Gittins (1979), Varaiya et al. (1985), Mandelbaum (1987)

El Karoui and Karatzas (1993), Kaspi and Mandelbaum (1998).

$(1-\beta)E_x \sum_{n=0}^{\tau-1} \beta^n = 1 - E_x\beta^\tau = P_x(Z_\tau = e) \equiv Q^\tau(x)$

GI $\gamma(x) = (1-\beta)\alpha(x)$, where

$$\alpha(x) = \frac{E_x \sum_{n=0}^{\tau-1} \beta^n c(Z_n)}{E_x \sum_{n=0}^{\tau-1} \beta^n} = \sup_{\tau>0} \frac{R^\tau(x)}{Q^\tau(x)},$$

$Q^\tau(x)$ is a probability of *termination*.

Four indices : GI $\gamma(x) = (1-\beta)\alpha(x)$, GGI (Mitten 1960); Kathehakis and Veinot (1987) $h(x)$; Whittle $w(x)$

Theorem 4. *If* $\beta = const$ *then* $\frac{\gamma(x)}{1-\beta} = \alpha(x) = h(x) = w(x)$.

But if $\beta(x)$ is not a constant then the *proportionality* of $\gamma(x)$ and $\alpha(x)$ as functions of x *completely disappears.*

Mitten (1960); Granot and Zuckerman (1991); Denardo, Rothblum, and Van der Heyden, (2004)

Presman and Sonin (2005) *A (Gittins) Index Theorem for Randomly Evolving Graphs* in: *The Shiryaev Festschrift*, Kabanov, Y; Lipster, R; Stoyanov, J (Eds.), Springer, 2006,

Sonin *A Generalized Gittins Index for Markov Chain and its Recursive Calculation.* appear in *Statistics and Probability Letters*, (2008).

Elementary (Mitten) model of independent trials

$$\begin{array}{cccccc} \uparrow & \uparrow & \cdot & \cdot & \cdot & \uparrow \\ e_1 & e_2 & \cdot & \cdot & \cdot & e_n \end{array}$$

Two possible outcomes in each trial, **"continuation"** with probability p_i in i-th trial, and **"termination"** with probability q_i.

A decision maker (DM) can choose an order in which to conduct (test) the trials. Each trial can be tested *only once*. The test of i-th trial brings a **reward** r_i. In the case of "termination" the testing has to be *terminated*. In the case of "continuation" the DM may continue testing or quit. The goal of DM is to select the optimal order to **maximize the expected total reward**.

The formulation is equivalent to a formulation where DM has to pay an amount c_i in advance, obtains a_i with probability p_i, and b_i with probability q_i, and $r_i = -c_i + a_i p_i + b_i q_i$.

A rather simple proof shows that the optimal strategy has a remarkable simple structure and is based on an **index** α calculated for *each trial* e_i, $\alpha(e_i)$ equal to **expected profit divided by probability of termination**, i.e.

$$\alpha(e_i) = \frac{r_i}{q_i}.$$

The optimal strategy has the following form: test the trials with positive index in the order of decreasing. If all trials must be tested then all they should be tested in the above order.

This problem is a reformulation of a “least cost testing sequencing” problem solved independently by a few authors in 1960, in particular by Mitten.

Indices $h(x)$ and $w(x)$.

Kathehakis and Veinot (1987) Markov Decision model

$M = (X, x_1, A(x), P, c(x), \beta(x))$

$A(x) = \{ c = \textit{continue}, \mathrm{r} = \textit{return to } x_1 \}$

$h(x) = \sup_\pi E_x^\pi \sum_{n=0}^\infty r(Z_n, A_n)$,

where $r(x, a) = c(x)$ if $a = c$, $r(x, a) = 0$ if $a = r$

$$h(x_1) = \sup_{\tau > 0} E_x [\sum_{n=0}^{\tau - 1} c(Z_n) + h(x_1)].$$

Whittle Retirement Process

family of OS models $M(k) = (X, P, c(x), k, \beta(x))$,

$g(x|k) = k$ if $x \neq e, g(x|k) = 0$ if $x = e$,

value function $v(x, k) = \sup_{\tau \geq 0} E_x[\sum_{n=0}^{\tau-1} c(Z_n) + g(x|k)]$

$w(x) = \inf\{k : v(x, k) = k\}$

Theorem 5. *The three indices coincide, i.e.* $\alpha(x) = h(x) = w(x)$.

Apply SE algorithm:
$g(x) - Fg(x) = k - (c(x) + \beta(x)k) = (1 - \beta(x))(k - d(x))$, where function $d_1(x) = \frac{c(x)}{1-\beta(x)}$, number $d = \max_{x \in X} d(x)$, set $D = \{x : d(x) = d\}$.

Lemma 4 *Let* $M(k)$ *be the corresponding Whittle OS model. Then* $\alpha(x) = d$ *for* $x \in D$, and $\alpha(x) < d$ *for all* $x \in X \setminus D$.

Algorithm to calculate $\alpha(x)$

model $M_1 = (X_1, P_1, c_1(x), \beta_1(x))$, function $d_1(x) = \frac{c_1(x)}{1-\beta_1(x)}$,

number d_1, set $D_1 : \alpha(x) = d_1$ for all $x \in D_1$;

eliminate D_1, recalculate $P_1 \to P_2, c_1(x) \to c_2, \beta_1(x) \to \beta_2$,

obtain $d_2(x)$, d_2, set D_2. Then by lemma 4 $\alpha(x) = d_2$ for all $x \in D_2$ and so on.

Example 3. Let our reward model has $X_1 = \{1, 2, 3, x_*\}$, with $c_1(1) = 3, c_1(2) = 2, c_1(1) = 1$, and $\beta(x) = .9$ for all $x \neq x_*$ and corresponding transition matrix $P = P_1$ is

$$P_1 = \begin{array}{|c|c|c|c|}\hline .3 & .3 & .3 & .1 \\ \hline .45 & .3 & .15 & .1 \\ \hline .1 & .5 & .3 & .1 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}\,, \; P_2 = \begin{array}{|c|c|c|}\hline .5 & .34 & .16 \\ \hline .54 & .34 & .12 \\ \hline 0 & 0 & 1 \\ \hline \end{array}\,, \; P_3 = \begin{array}{|c|c|}\hline .71 & .29 \\ \hline 0 & 1 \\ \hline \end{array}$$

Then $d_1 = c_1(1)/(1 - \beta) = 30, D_1 = \{1\}$, and by Lemma 2 $\alpha(1) = 30.$ Therefore, eliminate state 1 on a first step and, applying formulas (2) and (3), obtain new transition matrix P_2 and function $c_2(x)$ for a state space $X_2 = \{2, 3, x_*\}$; $c_2(2) = 3.93, c_2(3) = 1.43.$ Therefore $d_2 = c_2(2)/(1 - \beta_2(2)) = 3.93/0.16 = 24.56 = \alpha(2), D_2 = \{2\}$ and on the second step state 2 is eliminated, and we obtain matrix P_3, and $c_3(3) = 5.63.$ Therefore $\alpha(3) = c_3(3)/(1 - \beta_3(3)) = 5.63/0.29 = 19.41.$

Note that though we started in this example from a constant survival function $\beta(x)$, after the first step we deal with variable $\beta_i(x)$ for $i > 1$. The classical GI for this model $\gamma(x) = (1 - \beta)\alpha(x) = .1\alpha(x)$.

Three abstract optimization problems: There is an index set U, and $A = \{a_u\}$ and $B = \{b_u\}$ be two sets indexed by the elements of U. Suppose that an assumption AB holds,

$$a_u \le a < \infty, \quad 0 < b \le b_u \le 1 < \infty.$$

Problem 1 A. Restart pr-m. Find solution(s) of the equation

$$h = H(h) = \sup_{u \in U}[a_u + (1 - b_u)h]) \tag{1}$$

The interpretation of this problem is as follows. There is a set of "buttons" U. A DM can select one of them and push. She obtains a reward a_u and with probability b_u the game is over, with complimentary probability $1 - b_u$ she can select any button again. Her goal is to maximize the total (undiscounted) reward. It can be easily proved that her value satisfies the equation above.

[in Mitten problem each button can be used only once.]

Problem 2 A. Ratio (cycle) pr-m. Find

$$\alpha = \sup_{u \in U} \frac{a_u}{b_u} \tag{2}$$

Problem 3 A. A parametric family of Retirement pr-ms.

Find w, defined as follows: given parameter $k, -\infty < k < \infty$, let

$$v(k) = k \vee H(k), \quad w = \inf\{k : v(k) = k\}. \tag{3}$$

Theorem 1 A. a) $\alpha < \infty$ *and is a (unique) solution of equation* (1)

b) $\alpha = h = w$.

c) *The optimal index* u, *if exists, or an optimizing sequence* u_n *coincide for all three problems.*

Open Problems. Plans

- more examples when SEA is efficient
- analog of SEA in continuous time/space
- games
- Multiple restart problems
- Multidimensional Abstract Optimization ?

Bonus Slides

Let us consider the following simple random experiment; we flip a *coin*, we toss a *die*. Then our sample space consists of 12 outcomes each having a probability 1/12. This experiment is used in many textbook as an illustration of a concept of independent events.

Question 1. How many different *pairs* (A, B) *of independent events* are there ?

Question 2. How many different *tuples of independent events* $(A_1, A_2, ..., A_k)$ are there ?

The answers for both questions, numbers N_1 and N_2 are a bit puzzling.

Answers

The first number, $N_1 = 888,888.$

The second, $N_2 = 30,826,488.$

Thank you for your attention !