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ORIGINAL ARTICLE

The existence and uniqueness of Nash equilibrium point in an *m*-player game "Shoot later, shoot first!"

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Abstract We consider the following "silent duel" of m players with a possible economic interpretation. Each player has one "bullet", which she can shoot at any time during the time interval $[0, 1]$. The probability that the i-th player hits the "target" at moment t is given by an increasing accuracy function $f_i(t)$. The winner is the player who hits the target first. Under natural assumptions on the functions $f_i(t)$ we prove the existence and uniqueness of a Nash equilibrium point in this game, and we provide an explicit construction of this equilibrium. This construction allows us to obtain exact solutions for many specific examples. Some of them are presented.

Keywords Duel of m players · Nash equilibrium

1 Introduction

We consider the game of m players with a possibly broad interpretation, but first we formulate it as follows. Each of m players is initially located at a distance from a "target". It can be a common target or each player has her/his own target, or in the case of two players, each serves as a target for the other. At the moment $t = 0$ each player starts to move towards the target. Each player

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has one “bullet”, which she/he can shoot at any time during the time interval $[0, 1]$. The probability that the ith player *hits the target* at moment t is given by an *accuracy function* $f_i(t)$, where $f_i(t)$ is a continuous, (strictly) increasing function, $0 \le f_i(t) \le 1$, and $f_i(0) = 0$. The *winner* of the game is the player who *hits the target first*. If two or more players hit the target at the same time, the winner is determined by a lottery (*draw rule*) among these players. The rules of the lottery might depend on time, the probability that nobody in a lottery is a winner or that there is more than one winner might be positive, and the probability of winning the lottery is less than one for each participant of the lottery.

For example, if $k>1$ players hit the target at some time then possible draw rules might be: “none of these k players is a winner”; “there is only one winner, and probability to be a winner for each of them equals to $1/k$”; “for each of these k players the result – to be a winner – is defined in an independent Bernoulli experiment”.

The players have no information about the actions of other players, and thus *their strategies are probabilistic measures* β_i *on the unit time interval* $[0, 1]$ describing their randomized selection of the moment of fire. We call this game “Shoot later, shoot first”, and because of the duel interpretation we will call a player “he” instead of the usual “she”.

This game has also a transparent economic interpretation as an attempt to model the situation when each of m decision makers has to make a “decision” during a fixed time interval. This decision can be correct or wrong, and the probability that this decision is correct is $f_i(t)$. The winner is the person who first makes a correct decision. A decision can be the launching of an enterprise of a new type, the acceptance of an offer or a bid, and so on. The launching of Amazon.com or Bill Gates’ leaving Harvard, come to mind, but probably there is even better and more applied interpretation of this game.

Denote by U the set of all players, the generic notation for the subset of U will be J. We denote by $|J|$ the number of elements in J. Our main result holds under the following assumption which essentially says that none of the players has a dominant *relative rate* of accuracy to hit the target. Let

$$r_i(t|J) := \frac{f_i(t)}{\left(\prod_{j\in J} f_j(t)\right)^{1/(|J|-1)}}, \quad i \in J, \;\; |J| \ge 2. \tag{1}$$

ND (no dominance) assumption. Functions $r_i(t) := r_i(t|U)$ are decreasing functions for every $i \in U$.

This assumption holds, for example, for any increasing functions $f_i(t)$ if $m = 2$ or any m if $f_i(t) = d_i f(t),\ i \in U$.

If $i \in J \subset J',\ |J|' > |J| \ge 2$, then direct calculations give

$$r_i(t|J) = r_i\left(t|J'\right)\left(\prod_{k\in J'\setminus J} r_k\left(t|J'\right)\right)^{1/(|J|-1)} = r_i(t)\left(\prod_{k\in U\setminus J} r_k(t)\right)^{1/(|J|-1)}. \tag{2}$$

The righthand side of (2) and ND-assumption imply that

$$r_i(t|J) \text{ decreases for any } J \subseteq U \text{ and } i \in J. \tag{3}$$

If all functions $f_i(\cdot)$ have derivatives $f_i'(t)$ then all functions $r_i(t|J)$ have derivatives $r_i'(t|J)$ on $(0,1)$. If some of $f_i(\cdot)$ have singular components, then they have a generalized (weak) derivatives $f_i'(t)$ (see, for example, Gel'fand and Shilov 1964). In the rest of the paper the upper index $'$ for a continuous function of bounded variation (in particular for $r_i(t|J)$) denotes the generalized derivative (density) of this function.

Denote by

$\beta \equiv (\beta_1, \ldots, \beta_m)$ a tuple of probabilistic measures (strategies), where β_i is a strategy for the player i, and $\beta_i(t) \equiv \beta_i([0,t])$ is the distribution function of β_i,

$\beta_{-i} \equiv \left(\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_m\right)$ the strategies of all players except for the i-th player,

$\varphi_i(\beta) \equiv \varphi_i(\beta_i, \beta_{-i})$ the *reward of i-th player,* i.e. his probability to be a winner when a tuple of strategies β is used.

We assume that a draw rule is fixed and we will *skip the indication to this rule, and sometimes to β and β_i* if it is clear from the context which β and β_i are considered.

As usual a tuple of strategies (probabilistic measures) $\beta = (\beta_1, \ldots, \beta_m)$ is called a *(Nash) equilibrium point (E-point)* if any player $i \in U$ can not increase his reward by deviating from his strategy at equilibrium if other players use their equilibrium strategies, i.e. $\varphi_i(\gamma, \beta_{-i}) \leq \varphi_i(\beta_i, \beta_{-i})$ for any strategy γ, $i \in U$.

In Presman and Sonin (1977) a theorem was proved, which we present below in a modified form as Theorem 1.

To formulate this theorem we need new notation. Let us consider a set $\mathcal{A}$ consisting of the vectors $\mathbf{a} \equiv (a_0, a_1, \ldots, a_m)$ such that $0 < a_0 < a_i \leq 1, 1 \leq i \leq m$, and at least two coordinates of $\mathbf{a}$ equal to 1. For any given vector $\mathbf{a} \in \mathcal{A}$ let us denote $\mathbf{b} = \mathbf{b}(\mathbf{a})$ a vector obtained from $\mathbf{a}$ by reordering its coordinates and removing all but one from each group of coinciding coordinates, so that $\mathbf{b} = (b_0, b_1, \cdots, b_s)$, $s = s(\mathbf{a})$, $1 \leq s \leq m-1$, $0 < b_0 = a_0 < b_1 < b_2 < \ldots < b_s = 1$. Denote $J_k = J_k(\mathbf{a}) = \left\{i : \ a_i \geq b_{k+1}\right\}$, $0 \leq k \leq s-1$, so that $J_0 = U$, $|J_{s-1}| \geq 2$. Note that, vice versa, a vector $\mathbf{b}$ and decreasing sets J_k, $0 \leq k \leq s-1$, with this property, uniquely define a vector $\mathbf{a}$.

Theorem 1 *Under the ND assumption the game "Shoot later, shoot first" has a unique Nash equilibrium point $\beta = (\beta_1, \ldots, \beta_m)$, which does not depend on a specific draw rule. This β has the following structure:*

(a) *there is a vector $\mathbf{a} = \mathbf{a}_* \in \mathcal{A}$ such that for every i the density of β_i equals to zero outside interval (a_0, a_i), and only one of the measures, say v, may have a discrete component, which is concentrated at $a_v = 1$;*
(b) *the density of $\beta_i, i \in U$, on the interval $\left(b_k, b_{k+1}\right)$ such that $b_{k+1} \leq a_i$, equals*

$$\beta_i'(t) = -\frac{h_{i,k}}{f_i(t)} \frac{r_i'(t|J_k)}{r_i\left(b_k|J_k\right)} > 0, \tag{4}$$

where $0 \le k \le s-1 \le m-2$, $\mathbf{b} = \mathbf{b}(\mathbf{a})$, $J_k = J_k(\mathbf{a})$, $s = s(\mathbf{a})$; $h_{i,0} = 1$, $h_{i,k} = h_{i,k}(\mathbf{b}) = \prod_{l=0}^{k-1} r_i\left(b_{l+1}|J_l\right)/r_i\left(b_l|J_l\right)$;

(c) *vector* $\mathbf{b}$ *and the sets* J_k, $0 \le k \le s-1$ *can be obtained recursively beginning with* $k = 0$ *using* (4) *and the condition that* $\beta_i(a_i) = 1$ *for all except possibly one player* v, *for whom* $a_v = 1$, $\beta_v(1_-) \le 1$;

(d) *the probability that player* i *wins is equal* $f_i(a_0)$, $i \in U$.

Thus all players start to "fire" at the same moment, i.e. the density of time of shooting becomes positive. They continue to fire continuously without interruption but generally cease fire at different moments except at least two players who fire until the last moment $t = 1$, and one of them may fire with positive probability at this moment. As a result the draw rule does not matter because the measures of all players except maybe one are continuous (do not have a discrete component), and therefore with probability one no two players will hit the target at the same moment.

Now we outline briefly the scheme of the proof of Theorem 1. In section 3 we formulate five lemmas and show that Theorem 1 follows from this lemmas.

In Lemma 1 we prove that only one measure from an E-point may have a discrete component, which is concentrated at point $t = 1$, and inside the interval $(0, 1)$ each measure is continuous and concentrated on a set, where reward function of this player attains its maximal value. *Reward function* $g_i(t|\beta_{-i})$ of i-th player at t is the probability that i-th player is a winner if he shoots at t, and other players use strategies from tuple β, $i \in U$. We prove also that if a tuple of strategies satisfies these properties then it defines an E-point.

Using Lemma 1 we prove Lemma 2, which says that for any E-point there exists a value a_0, $0 < a_0 < 1$, such that: (a) all reward functions do not attain the maximal value on $0 < t < a_0$; (b) for any $t > a_0$ there are at least two players such that reward functions of these players attain maximal value at t; (c) in any right neighborhood of any t, $a_0 < t < 1$, there exists an *interval* such that for some (at least two) players their reward functions attain maximal value on this interval, and all other players have zero densities on this interval.

Given a tuple of strategies, we call a subinterval of $(0, 1)$ a *proper* one if for some (at least two) players the reward functions are constant on this subinterval, and all other players have zero densities on this subinterval. We call the players of the first kind *active* players and we will show that they have positive densities. All other players are called nonactive. Thus, the interval from point (c) of Lemma 2 is a proper one.

In Lemma 3 we obtain an explicit formula (4) for the density of a measure $\beta_i'(t)$ on a proper interval for each active player. These measures are defined up to constants ($h_{i,k}$ in (4)) which can be found if information is available about previous intervals.

Using Lemmas 2 and 3 we show in Lemma 4 that any E-point must have a general structure described in points (a) and (b) of Theorem 1, i.e. an E-point $\mathbf{a} \in \mathcal{A}$ consists of a finite number of proper intervals, where at the end of each interval one or may be a few active players became nonactive forever, and at least two players remains active till $t = 1$, and one of them may have a positive mass at $t = 1$.

Note that knowing the initial point a_0 of an E-point, i.e. the first coordinate of the corresponding $\mathbf{a}$, we could have constructed $\mathbf{a}$ as follows. By Lemma 4 all players are active on the first proper interval and hence each density $\beta_i'(t), i \in U$, is defined by (4). Then the right end b_1 of the first interval can be found as the minimal point where the first time one or maybe a few players gain the full measure on this interval, i.e. from the condition $\beta_i((a_0, b_1)) = 1$ for some i. All such players become nonactive and for them $a_i = b_1$. All other players remain active, and using for them formula (4) on the interval (b_1, b_2), we can find similarly the right end b_2 of the second proper interval and so on. After finite number of steps we obtain vector $\mathbf{a}$ and all strategies.

But the value a_0 is unknown. To specify this value we will use the recursive construction described above for an *arbitrary* b, $0 < b < 1$, taking this b as an initial point of the first proper interval. As a result we obtain for an arbitrary b a vector $\mathbf{a}(b)$ and the strategies $\beta_i(\cdot|b)$ such that corresponding measures equal zero outside of $(b, a_i(b)]$, and inside of $(b, a_i(b))$ they have densities defined by (4). For an arbitrary b we can have three situations.

The first situation occurs when at the end of the recursive process all the players except possibly one have gained the full measure before point 1 was reached. In this case $\mathbf{a} \notin \mathcal{A}$ and according to Lemma 4, such $\beta(\cdot|b)$ can not be an E-point. As we will see later this means that $b < a_0$, where a_0 is from Theorem 1.

The second situation occurs when at the end of the recursive process more than one player still have not gained the full measure on $(b, 1)$. Then the corresponding strategies (more than one) should have discrete components at point 1 and, according to Lemma 1, such $\beta(\cdot|b)$ can not be an E-point too. This means that $b > a_0$.

The third situation occurs when all players, except possibly one, gain the full measure on $(b, 1)$, and at least two players do not gain the full measure before point 1 is reached. According to the second part of Lemma 1, for any such b the tuple $\beta(\cdot|b)$ is an E-point, i.e. $b = a_0$.

The uniqueness is proved in Lemma 5, where we show that our construction is monotonic. After that we show that Theorem 1 follows from Lemmas 1–5.

The game "Shoot later, shoot first" formally belongs to the class of silent one-action duels in the terminology of Karlin (1959). That volume is a compendium of many classical results in games of timing, which were intensively developed by a group of prominent specialists in game theory, stochastic control and social sciences working in 1950s and 1960s in the RAND corporation (Blackwell, Girschick, Karlin, Shapley and many others). A thorough survey of zero-sum games of timing with two players and a review of more general results are given by Radzik (1996). A solution of the game "Shoot later, shoot first" for $m = 2$ is given by Garnaev (2000). He was not aware of our paper Presman and Sonin (1977). In his book for the case of two players he also solved the case of noisy duel when a player knows when the other player is shooting and the case of nosy-silent duel when one of the players has a silent gun. The duels with random termination time were introduced in Teraoka (1983) and Sakaguchi (1987). The generalizations of these results are also discussed in Garnaev (2000).

The games with $m > 2$ are traditionally difficult to solve and most results on timing problems are concerned with rather particular cases. We hope that methods developed in our paper will help to solve for example games described in Garnaev (2000) for $m > 2$.

Our initial motivation for this problem was an attempt to generalize the Classical Secretary Problem to the case when selection is made not by one but by many players. The paper Presman and Sonin (1977) was published in Russian in a collection of papers of CEMI (CEMI is an abbreviation for Central Economics & Mathematics Institute of the Russian Academy of Sciences, which was at that time a kind of Soviet civic analog of the RAND corporation in the USA). The paper was never translated into English. The proof was very cumbersome and difficult and had minor gaps. At the same time the problem provides a relatively rare case when a Nash equilibrium point is unique and can be described in an explicit form. The goal of this paper is to present a simplified and streamlined proof of Theorem 1, to describe an algorithmic approach to construct the equilibrium point, and present some examples. We also hope that maybe new original interpretation will appear as a result of this publication. Note that the first results of this type were obtained in Sonin (1976) and Presman and Sonin (1975).

The structure of the paper is as follows. In section 2 we present three examples which demonstrate how to find the E-point using Theorem 1. In section 3 we present the essential elements of the proof of Theorem 1. Technical statements are given in the Appendix.

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2 Examples

Example 1 Case $m = 2$. In this case $r_1(t) = 1/f_2(t),\ r_2(t) = 1/f_1(t)$, and, according to Theorem 1, the continuous part of E-point is given by vector $\mathbf{a}_* = (a_0, 1, 1)$ and by densities: $\beta_i(t|b) = 0$ for $0 < t < b,\ \ i = 1, 2,$

$$\beta_1'(t|b) = \frac{f_2(b)}{f_1(t)}\frac{f_2'(t)}{(f_2(t))^2}, \quad \beta_2'(t|b) = \frac{f_1(b)}{f_2(t)}\frac{f_1'(t)}{(f_1(t))^2}, \quad \text{for} \quad b < t < 1, \tag{5}$$

with $b = a_0$. To determine a_0, we consider densities defined by (5) for arbitrary $b,\ 0 < b < 1$. Let $\beta_i(1|b) = \int_0^1 \beta_i'(t|b)dt = \int_b^1 \beta_i'(t|b)dt,\ i = 1, 2$. In Proposition 6 in the Appendix for a more general situation we will show that $\beta_i(1|b),\ i = 1, 2,$ decreases from $+\infty$ to 0 as b increases from 0 to 1. So, there exists b_i such that $\beta_i(1|b_i) = 1,\ i = 1, 2$. Denote $b_* = \max(b_1, b_2),\ \mu_i \equiv 1 - \beta_i(1|b_*)\ i = 1, 2$. It follows from the monotonicity of $\beta_i(1|b)$ that $0 \le \mu_i < 1,\ \mu_1\mu_2 = 0$. By Theorem 1, $a_0 = b_*$, and measure $\beta_i(\cdot)$ has a jump μ_i at $t = 1$.

If $f_1(t) = t$, $f_2(t) = t^\alpha$, then it is not difficult to check that $\beta_1'(t) = \alpha b_*^\alpha / t^{2+\alpha} = \alpha b_*^{\alpha-1} \beta_2'(t)$ for $b_* < t < 1$. If $\alpha \geq 1$ then b_* satisfies the equation $1/b = (b^\alpha + \ (\alpha + 1/\alpha))$, and the second player has mass $1 - \alpha^{-1} b_*^{1-\alpha}$ at $t = 1$. If $\alpha \leq 1$ then b_* satisfies the equation $b^\alpha\ (b + \alpha + 1) = 1$, and the first player has mass $1 - \alpha b_*^{\alpha-1}$ at $t = 1$. In case $\alpha = 2$ we have: $b_* \approx 0,55$, probability φ_1 that the first player is a winner equals $\approx 0,55$, $\varphi_2 \approx 0,30$, probability that nobody is a winner equals $\approx 0,15$, the second player has mass $\approx 0,09$ at $t = 1$, $\beta_i'(t) = c_i/t^4$ for $t > b_*$, where $c_1 \approx 0,60$, $c_2 \approx 0,56$.

Example 2 $f_i(t) = f(t)d$ for all $i = 1, \ldots, m$, where $f(t)$ is an increasing function, $f(1) = 1$, $f(0) = 0$, $d \leq 1$. In this case $r_i(t) = f(t)d/(f(t)d)^{m/(m-1)} = (f(t)d)^{-1/(m-1)}$. Since $r_i(t)$ does not depend on i and $\beta_i(a_i) = 1$ for all except possibly one player, point 1 (b) of Theorem 1 implies that $\beta_i(t)$ does not depend on i, and therefore according to Theorem 1, the E-point is defined by $\mathbf{a}_* = (a_0, 1, \ldots, 1)$ and by density $\beta(t|b) = \beta_i(t|b)$ for $0 < t < 1$, $i \in U, \beta(t|b) = 0$ for $0 < t < b$,

$$\beta'(t|b) = \frac{1}{f(t)d} \frac{1}{m-1} \left(\frac{f(b)}{f(t)}\right)^{1/(m-1)} \frac{f'(t)}{f(t)} \quad \text{for } b < t < 1, \tag{6}$$

with $b = a_0$. To determine a_0, we consider density defined by (6) for an arbitrary b, $0 < b < 1$. Let $\beta(1|b) = \displaystyle\int_0^1 \beta'(t|b)dt = \int_b^1 \beta'(t|b)dt$, so that $\beta(1|b) = 1/mf(b)d - (f(b))^{1/(m-1)}/md$. It can be checked that $\beta(1|b)$ decreases from $+\infty$ to 0 as b increases from 0 to 1. So, there exists b_* such that $\beta(1|b_*) = 1$. It follows from Theorem 1 that $a_0 = b_*$.

Note that if the E-point for $f(t) \equiv t$ is given by a_0, then the E-point for an arbitrary $f(t)$ is given by $\tilde{a}$, where $f(\tilde{a}) = a_0$. The value a_0 satisfies the equation $b^{-1/(m-1)} \left(b^{-1} - md\right) = 1$. Obviously, $(md+1)^{-1} < a_0 < (md)^{-1}$. Moreover, it is easy to check that $\left(md + (md)^{-1/(m-1)}\right)^{-1} < a_0 < \left(md + (md+1)^{-1/(m-1)}\right)^{-1}$. For $d = 1, m = 2$ we have $a_0 = \sqrt{2} - 1 \approx 0,414$. For $d = 1, m = 3$ we have $a_0 \approx 0,283$.

Example 3 $f_i(t) = d_i f(t)$, $0 < d_i \leq 1$, for $i = 1, \ldots, m$, where $f(t)$ is a strictly increasing function such that $f(0) = 0$, $f(1) = 1$. We assume that there are n groups $V_1, \ldots, V_n$ of players with the same d_i inside each group, $1 \leq n \leq m$, i.e. $d_i = v_k$ if $i \in V_k$, where $0 < v_1 < \cdots < v_n$.

In this case $r_i(t|J)/r_i(b|J) = (f(b)/f(t))^{1/(|J|-1)}$ for $0 < b < t < 1$. According to Theorem 1, continuous part of the E-point is defined by $\mathbf{a} = \mathbf{a}_* \in \mathcal{A}$ and by densities $\beta_i'(t)$, where: $\beta_i'(t) = 0$ for $t \notin (a_0, a_i)$,

$$\beta_i'(t) = \frac{h_k}{|J_k| - 1} \frac{f'(t)}{f(t)} \frac{1}{d_i f(t)} \left(\frac{f(b_k)}{f(t)}\right)^{1/(|J_k|-1)} \tag{7}$$

for $t \in (b_k, b_{k+1})$ and for all k, $0 \le k \le s$, such that $b_{k+1} \le a_i$, where $h_0=1$, $h_k = \prod_{l=0}^{k-1} \left(f(b_l)/f\left(b_{l+1}\right)\right)^{1/(|J_l|-1)}$ for $k \ge 1$. Here $\mathbf{b}=\mathbf{b}(\mathbf{a})$, $J_k = J_k(\mathbf{a})$, $s = s(\mathbf{a})$, and $\beta_i(1) < 1$ for at most one i.

Let $U_0 = U$, $U_k = U_{k-1} \setminus V_k$, $e_k = v_k \sum_{l=k}^{n} |V_l| + \sum_{l=1}^{k-1} v_l |V_l|$, $1 \le k \le n$. Note that $e_{k+1} > e_k$. Let us define functions $b_k(b)$, $k = 1, \ldots, n$, as follows

$$f\left(b_k(b)\right) = f(b)\left(1 - e_k f(b)\right)^{-\frac{|U_{k-1}|-1}{|U_{k-1}|}} \prod_{l=1}^{k-1} \left(1 - e_l f(b)\right)^{-\frac{|V_l|}{|U_l||U_{l-1}|}}. \tag{8}$$

It is easy to see that on the interval $\{b : e_k f(b) < 1\}, 1 \le k \le n$, the righthand side of (8) increases from 0 to $+\infty$, therefore there exist unique p_k such that $b_k(p_k) = 1$, and $p_{k+1} < p_k$. In the Appendix we provide the basic construction of the mappings $\mathbf{a}(b)$ and $\beta(\cdot|b)$ and show that: 1) if $i \in V_k$ and either $1 \le k \le n-1, |V_n| = 1$, or $1 \le k \le n, |V_n| > 1$, then $a_i(b) = b_k(b)$ for $0 < b < p_k$ and $a_i(b) = 1$ for $p_k \le b < 1$, $a_m(b) = a_{m-1}(b)$; 2) continuous part of $\beta_i(t|b)$, $1 \le i \le m$, is given by (7) where b_k are the coordinates of vector $\mathbf{b}(\mathbf{a}(b))$. Let $b_* = p_s$, where $s = n-1$ if $|V_n| = 1$, $s = n$ if $|V_n| > 1$. Theorem 1 implies that $\beta(\cdot|b_*)$ is an E-point. Player m has mass $\left(d_m - d_{m-1}\right)/d_m$ at point 1 for E-point.

Note that if $\mathbf{a}$ is the E-point for $f_i(t) = d_i t$, then for any function $f(t)$ the E-point corresponding to $f_i(t) = d_i f(t)$ is given by $\widetilde{\mathbf{a}}$, where $f(\widetilde{a}_i) = a_i$ for $i = 1, \ldots, m$. It is easy to check that for example in case $m = 3$, $d_1 = 1/3$, $d_2 = 1/2$, $d_3 = 1$, we have $e_1 = 1, e_2 = 4/3$, and $a_0 = b_*$ satisfies equation $1 = b\,(1 - 4b/3)^{-1/2}\,(1-b)^{-1/6}$, so that $a_0 \approx 0,51$. Further, $a_1 = b_*(1-b*)^{-2/3}$ $\approx 0,82$, $a_2 = a_3 = 1$, and the third player has mass $1/2$ at $t = 1$. Probabilities to be a winner are $\varphi_1 \approx 0,17$, $\varphi_2 \approx 0,255$, $\varphi_3 \approx 0,51$, the probability that nobody is a winner is $\approx 0,065$.

3 Proof of Theorem 1

Given a tuple of strategies β, denote by $h_i(t|\beta_i)$ the probability that i-th player *will not hit the target before t using his strategy* β_i, i.e.

$$h_i(t|\beta_i) = 1 - \int_0^{t-0} f_i(s)\,d\beta_i(s), \tag{9}$$

$g_i\left(t|\beta_{-i}\right)$ a *reward function of ith player at t*, i.e. the probability that i-th player is a winner if he shoots at t, and other players use strategies from a tuple β, $i \in U$.

Then the reward of ith player, i.e. his probability to be a winner, is

$$\varphi_i(\beta) = \int_0^1 g_i\left(t|\beta_{-i}\right) d\beta_i(t), \quad i \in U. \tag{10}$$

Note, that if all measures β_i have no discrete components on $(0,1)$ then obviously

$$g_i\left(t|\beta_{-i}\right) = f_i(t)\prod_{j\neq i} h_j\left(t|\beta_j\right), \quad i \in U. \tag{11}$$

The expression for $g_i\left(t|\beta_{-i}\right)$ in general case will be discussed later in the Appendix, but only formula (11) will be used further.

Our game is defined as a game in normal form by formula (10), where $\beta_i, i \in U$, is a nonnegative measure on $(0,1]$ with distribution function (cumulative measure) $\beta_i(t), i \in U$, satisfying also the normalizing condition

$$\beta_i(1) = 1, \quad i \in U. \tag{12}$$

Given a tuple of strategies β, we use the simplified notation $g_i\left(t|\beta_{-i}\right) \equiv g_i(t)$ to define

$$m_i = \sup_{0\leq t\leq 1} g_i(t), \quad M_i = \{t : g_i(t) = m_i\}, \quad i \in U. \tag{13}$$

Lemma 1

1. *Let β be an E-point. Then for any draw rule*
   (a) *measures β_i for all players have no discrete components on* $[0,1]$ *except possibly one player whose strategy may have a positive measure at point* $t = 1$;
   (b) *measures β_i are concentrated on sets M_i, i.e. $\beta_i([0,1]) = \beta_i(M_i) = 1, i \in U$, and $\varphi_i(\beta) = m_i$;*
   (c) *corresponding functions $g_i(t)$ and $h_i(t)$ are continuous on* $[0,1)$ *and given by formulas* (11) *and* (9)
2. *Let a tuple of strategies β satisfies point* 1 (b). *Then β is an E-point.*

Remark 1 The meaning of statement (a) is intuitively clear. If a player i shoots with positive probability at some point t inside of the interval $(0,1)$ then for any other player it is not optimal to shoot on a sufficiently small interval $(t, t+\varepsilon)$, $\varepsilon > 0$, because this other player can increase his reward shooting instead a bit earlier than t. But if all other players do not shoot on this interval then for a player i it is better to shoot at $t+\varepsilon$ than at t, so a strategy of a player to shoot at t with positive probability can not be a part of an E-point. Statement (a) implies that if β is an E-point for some draw rule, then it is an E-point for any draw rule because by point (a) with probability one no two players shoot at the same time and thus a draw rule is never applied. Therefore for any E-point functions $g_i(t)$ are always given by (11). Statement (c) is a simple corollary of statement (a). Statement (b) is standard in game theory (see e.g. Lemma 2.2.1 in Karlin (1959)).

The proof of Lemma 1 is given in the Appendix.

Lemma 2 *Let β be an E-point. Then there exists a value a_0, $0 < a_0 < 1$, such that:*

(a) *if $0 < t < a_0$ then $t \notin M_i$ for any $i \in U$;*
(b) *for any $t > a_0$ there are at least two players $i, j \in U$ such that $t \in M_i \bigcap M_j$;*
(c) *for any t, $a_0 < t < 1$, and any $\varepsilon > 0$ there exists an interval (a,b) and a set of players $J = J(a,b)$ such that $|J| > 1$, $(a,b) \subset (t, t+\varepsilon)$, $(a,b) \subset M_j$ for all $j \in J$, $(a,b) \bigcap M_j = \emptyset$ for all $j \notin J$.*

The proof of Lemma 2 is given in the Appendix.

In section 1 we defined a notion of a proper interval as follows. For a given tuple of strategies an interval (a,b) is called a *proper* one and players from a set $J = J(a,b)$ are called *active* if on this interval

$$\frac{g_i(t)}{g_i(a)} = 1 \quad \text{for } \; i \in J; \qquad \frac{h_i(t)}{h_i(a)} = 1 \quad \text{for } \; i \notin J. \tag{14}$$

The next Lemma describes the main properties of a proper interval.

Lemma 3

1. *If, given a tuple of continuous on $[0,b)$ strategies, an interval (a,b) is a proper one and J is a set of active players, then*

$$\frac{h_i(t)}{h_i(a)} = \frac{r_i(t|J)}{r_i(a|J)}, \quad \beta_i'(t) = -\frac{h_i(a)}{f_i(t)} \frac{r_i'(t|J)}{r_i(a|J)} > 0, \quad i \in J, \tag{15}$$

$$\frac{g_i(t)}{g_i(a)} = \frac{r_i(t|J)}{r_i(a|J)}, \;\; \beta_i'(t) = 0 \;\; i \notin J. \tag{16}$$

2. *If, given a tuple of continuous on $[0,b)$ strategies, the right equalities in (15) and (16) hold on (a,b), then (a,b) is a proper interval and J is the set of active players.*
3. *An interval (a,b) is a proper one and J is a set of active players for an E-point iff $(a,b) \subset M_i$ for all $i \in J$, and $(a,b) \bigcap M_i = \emptyset$ for all $i \notin J$.*

The proof of Lemma 3 is given in the Appendix. Note that formula (4) is a particular case of the right equalities in (15) and (16), and $h_{i,k} = h_i(b_k)$.

It follows from (3) and (16) that the reward function of a nonactive player decreases on a proper interval. This key property along with Lemmas 1 and 2 allows to prove the following Lemma.

Lemma 4 *Let β be an E-point. Then there exists a vector $\mathbf{a} \in \mathcal{A}, \mathbf{a} \equiv (a_0, a_1, \ldots, a_m)$, such that continuous part of β is given by (4) and $\beta_i'(t) = 0$ for $t \notin (a_0, a_i)$, $i \in U$.*

The proof of Lemma 4 is given in the Appendix.

Let $\mathcal{A}_1$ be a set of vectors $\mathbf{a} \equiv (a_0, a_1, \ldots, a_m)$ such that $0 < a_0 < a_i \le 1$, $1 \le i \le m$, and at least two coordinates of $\mathbf{a}$ have maximal value (not necessary equal to 1). It is obvious that $\mathcal{A} \subset \mathcal{A}_1$.

It was mentioned in Section 1 that to specify $\mathbf{a}_*$ we have to consider a mapping $b \to \beta(\cdot|b)$ from the interval $(0,1)$ to the set of possible strategies, and accompanying mapping $b \to \mathbf{a}(b)$ from the interval $(0,1)$ to the set $\mathcal{A}_1$. Below we give the details of the construction of these mappings. We call it *basic construction*.

We set $\beta_i(t|b) = 0$ for $0 \le t \le b$, $i \in U$, and consider b as the initial point of the first proper interval on which formula (4) with $k = 0$, $J_0 = U$ is valid, i.e. we set $a_0(b) = b$. If for each $i \in U$ the corresponding measure of the interval $(0,1)$ is less or equal to 1, then we set $V_1 := V_1(b) = U$, $a_i(b) = b_1 = 1$ for all $i \in J_0$ and add mass $1 - \beta_i(1_-|b)$ to the point $t = 1$, $i \in U$. In this case our construction of $\beta(\cdot|b)$ and $\mathbf{a}(b)$ is completed.

Otherwise, there exist $b_1 = b_1(b) < 1$ and the set $V_1 := V_1(b)$ such that all players from V_1 gain full measure on the interval (b, b_1), and all players from $J_1 = J_1(b) := J_0 \setminus V_1$ have on this interval measure less than 1. In this case we set $a_i(b) = b_1$ for $i \in V_1$ and if J_1 is empty, then our construction of $\mathbf{a}(b)$ and measures $\beta_i(\cdot|b)$ for all players is completed. If $|J_1| = 1$ then we set $a_i(b) = b_1$ also for i from J_1 and add mass $1 - \beta_i(b_{1-}|b)$ to the point $t = b_1$ for this player. Thus our construction of $\beta(\cdot|b)$ and $\mathbf{a}(b)$ is completed again.

If $|J_1| \ge 2$ we have to continue our construction for the players from J_1 on the interval $(b_1, 1)$. This means that we consider b_1 as the initial point of the second proper interval on which formula (4) with $k = 1$, $J_1 = J_1(b)$ is valid. Now we define $b_2 := b_2(b)$, V_2 and J_2 in the same way as we have defined b_1, V_1, and J_1 for $k = 0$ and either complete our construction (if $b_2 = 1$ or $|J_2| < 2$) or consider b_2 as the initial point of the third proper interval. After at most $(m-1)$ steps, say $s = s(b)$, we complete the construction of $\beta(\cdot|b)$, and $\mathbf{a}(b)$.

Let

$$\overline{a}(b) = \max_{i \in U} a_i(b). \tag{17}$$

Our construction implies: $\mathbf{a}(b) \in \mathcal{A}_1$; each measure $\beta_i(\cdot|b)$ is concentrated on $(b, a_i(b)]$, $i \in U$; all measures except possibly one, say v, are continuous on $(0,1)$, and $\beta_v(\cdot|b)$ has a discrete component inside of $(0,1)$ iff $v \notin \bigcup_{k=1}^{s} V_k$, in this case $a_v = \overline{a}(b) < 1$ and there exists $u \ne v$ such that $a_u = \overline{a}(b)$.

It follows from the construction that if b is close to 1, then $\mathbf{a}(b) = (b, 1, \ldots, 1)$ and $\beta_i(1_-|b) \to 0$ as $b \to 1$. So, it is natural to consider points

$$s_i = \sup\{b : \beta_i(1_-|b) = 1\},\ i = 1, \ldots, m, \quad b_* = \min_i s_i, \tag{18}$$

where supremum over empty set equals to 0. The following lemma describes the monotonicity properties of functions $a_i(b)$ and $\beta_i(1_-|b)$.

Lemma 5

(a) $0 < b_* < 1$;
(b) *functions* $a_i(b)$ *are continuous on* $(0,1)$, *strictly increasing on the interval* $(0,s_i)$ *when b is increasing, and equal* 1 *on* $[s_i,1]$, $i \in U$;
(c) *functions* $\beta_i(1_-|b)$, $i \in U$, *are continuous on* $(0,1)$ *for all except possibly one player, say v, equal* 1 *on* $(0,s_i)$, *decreasing on the interval* $(s_i,1)$ *when b is increasing,* $\beta_v(1_-|b)$ *may have a negative jump at* $b = b_*$;
(d) *if* $0 < b < b_*$ *then* $\overline{a}(b) < 1$; *if* $b_* \le b < 1$ *then* $\mathbf{a}(b) \in \mathcal{A}$.

The proof of Lemma 5 is given in the Appendix.

Let us show that Theorem 1 follows from Lemmas 1–5. First we show that $\beta(\cdot|b)$ is not an E-point for $b \neq b_*$. If $0 < b < b_*$ then $\overline{a}(b) < 1$ by point (d) of Lemma 5, i.e. $\mathbf{a}(b) \notin \mathcal{A}$, and hence by Lemma 4 $\beta(\cdot|b)$ is not an E-point. If $b = b_*$ then according to the definition of b_* and point (d) of Lemma 5 there exist u and v such that $a_u = a_v = 1$, $\beta_u(1_-|b) = 1$, $\beta_v(1_-|b) \le 1$, $s_u = s_v = b_*$. Then, if $b > b_*$, point (c) of Lemma 5 implies that $\beta_u(1_-|b) < 1$, $\beta_v(1_-|b) < 1$, i.e. both measures have positive masses at $t = 1$. By point (a) of Lemma 1, $\beta(\cdot|b)$ is not an E-point for such b also.

Now let us show that $\beta(\cdot|b_*)$ is really an E-point. Since $\beta_i'(t|b_*) = 0$ for $0 < t < b_*$, $i \in U$, formulas (9) and (11) imply that $h_i(b_*) = 1, g_i(t) = f_i(t)$, $i \in U$ for $0 < t \le b_*$, and consequently $g_i(t)$ increases on $(0,b_*)$. By point 2 of Lemma 3, $g_i(t)$ is constant on $\left[b_*,a_i\right)$, and by the first equality in (16) and (3), $g_i(t)$ decreases on $(a_i,1)$, $i \in U$. If all $\beta_i(\cdot|b_*)$ are continuous, then all $g_i(t)$ are continuous too, and, therefore, $M_i = \left[b_*,a_i\right]$, $m_i = f_i(b_*)$, $i \in U$. If $\beta_v(\cdot|b_*)$ has a positive mass at $t = 1$, then $a_v = 1$, $M_v = \left[b_*,1\right]$, $m_v = f_v(b_*)$, and $g_v(t)$ is continuous at $t = 1$. For $i \neq v$ in this case we have $g_i(1_-) > g_i(1)$, and for these players $M_i = \left[b_*,a_i\right]$ if $a_i < 1$, $M_i = [b_*,1)$ if $a_i = 1$, $m_i = f_i(b_*)$. Thus $\beta_i(\cdot|b_*)$ is concentrated on (b_*,a_i) for $i \neq v$, and $\beta_v(\cdot|b_*)$ is concentrated on $(b_*,1]$. Application of points 1 (b) and 2 of Lemma 1 completes the proof that $\beta(\cdot|b_*)$ is an E-point and satisfies all points of Theorem 1.

It remains to prove the uniqueness. If an E-point exists then by Lemmas 1 and 4 it satisfies points (a) and (b) of Theorem 1. The necessary condition $\beta_i(a_i) = 1$ and our construction imply that this E-point must coincide with $\beta(\cdot|b)$ for some b. But we proved that for all $b \neq b_*$ the tuple $\beta(\cdot|b)$ is not an E-point and for $b = b_*$ it is an E-point. Thus, $b_* = a_0$, where a_0 is from Theorem 1. This completes the proof of Theorem 1.

4 Appendix

Proof of Lemma 1

A draw rule $\mathbf{q}$ is defined as follows. For each $J \subset U$, $|J| \ge 2$, the following functions on $[0,1]$ are given: function $0 \le q(t,J) \le 1$ which defines the conditional probability that there is no winner at t under the condition that all players from

J and only they hit the target at t, and nobody was a winner on $[0,t)$; and for each $i \in J$ a function $0 \leq q_i(t,J) < 1$ which defines the conditional probability that player i is a winner under the same condition.

For fixed $\mathbf{t}$ and i denote $\mathbf{t}_{-i} = \left(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_m\right)$. Formally the game can be defined by kernels $K_i\left(t_i,\mathbf{t}_{-i}\right)$, where K_i is the gain of the player i, i.e. the conditional probability that player i is a winner under condition that all players shoot at times $\mathbf{t} = (t_1,\ldots,t_m)$. Then

$$\begin{aligned} g_i(t) &= g_i\left(t|\beta_{-i},\mathbf{q}\right) \\ &= \int_0^t \ldots \int_0^t K_i\left(t,\mathbf{t}_{-i}\right)\beta_1(dt_1)\ldots\beta_{i-1}\left(dt_{i-1}\right)\beta_{i+1}\left(dt_{i+1}\right)\ldots\beta_m(dt_m). \end{aligned} \tag{19}$$

Using the total probability formula one can write an explicit expression for $K_i\left(t_i,\mathbf{t}_{-i}\right)$ through the functions f_j, $q(t,J)$ and $q_i(t,J)$, but this expression is complicated and plays no role in the proof. To prove Lemma 1 we need only the fact that function K_i is bounded, measurable and that $g_i(t)$ can be represented as

$$g_i(t) = \left(g_{1i}(t) + g_{2i}(t)\right)h_{-i}(t), \tag{20}$$

where $h_{-i}(t)$ and $g_{ji}(t)$, $j = 1,2$ denote the following conditional probabilities under the condition that player i shoots at time t and other players use the strategies from the tuple β_{-i}:

$h_{-i}(t) := P\left(A_{-i}(t)\right)$, where $A_{-i}(t) = \{$there is no winner before $t\}$;

$g_{1i}(t) := P($no other player than i hits at t, player i hits at t (and hence wins) $|A_{-i}(t))$;

$g_{2i}(t) := P($ player i and at least one other player hit at t and i wins the lottery $|A_{-i}(t))$.

Let D_i be the set of points of discontinuity of β_i, $D_{-i} = \bigcup_{j\neq i} D_j$. Then, with the help of (19) and (20), the following properties can be derived:

$h_{-i}(t)$ is nonincreasing, continuous at $t \notin D_{-i}$, and has negative jump and is left continuous at $t \in D_{-i}$;

$g_{1i}(t) = f_i(t)$ for all $t \notin D_{-i}$, and hence is continuous there;

$g_{2i}(t) = 0$ for all $t \notin D_{-i}$, and maybe positive at $t \in D_{-i}$;

$g_{1i}(t) + g_{2i}(t) \leq g_i(t) \leq f_i(t)$ for all t.

Using these facts and the assumption that for every player $q_i(t,J) < 1$, we obtain the following proposition.

Proposition 1 *If β_{-i}, $i \in U$, is fixed then function $g_i(t)$ is continuous at $t \notin D_{-i}$, and*

$$g_i(t-0) > g_i(t), \quad g_i(t-0) > g_i(t+0) \ \textit{for}\ t \in D_{-i}, \tag{21}$$

$$g_i(t) = f_i(t)h_{-i}(t) > 0 \quad \textit{for}\ \ t \notin D_{-i},\ t > 0. \tag{22}$$

The next statement is similar to the known statement in game theory (see, for example, Lemma 2.2.1 in Karlin (1959)).

Proposition 2 *A tuple β is an E-point iff*

$$\beta_i([0,1]) = \beta_i(M_i) = 1, \qquad i \in U. \tag{23}$$

Proof Consider the sequence of embedded sets $M_i^n = \{t : g_i(t) > m_i - (1/n)\}$. By the definition of m_i, the sets M_i^n are not empty and $\cap_n M_i^n = M_i$. If β_i belongs to the E-point, then $\beta_i\left(M_i^n\right) = 1$, otherwise using (10) it is easy to show that β_i is not optimal. Since $\cap_n M_i^n = M_i$ we obtain that $\beta_i(M_i) = 1$. On the other hand, if for some tuple β and corresponding M_i one has (23), then (10) imply that the gain of each player equals to m_i, and therefore it equals to the supremum over all possible gains under fixed strategies of the other players, i.e. tuple β is an E-point.

Proposition 2 and formula (22) imply in particular that

$$D_i \subseteq M_i \text{ and } m_i > 0 \quad \text{ for all } i. \tag{24}$$

Proposition 3 *If a tuple β is an E-point, then functions $\beta_i(t)$, $i \in U$ do not have discrete components, with a possible exception of one player, who may have a positive measure at point $t = 1$, and $g_i(t)$ are given by* (11).

Proof Let us show first that $D_i \cap [0,1) = \emptyset$ for every $i \in U$. Since $f_i(0) = 0$, we have $g_i(0) = 0 < m_i$, and hence $\beta_i(\{0\}) = 0$. If $0 < t \le 1$ and $\beta_i(\{t\}) > 0$ then $t \in D_i \subseteq M_i$ by (24), and hence by the first inequality in (21) $t \notin D_{-i}$. Thus, all discrete components of all players are mutually singular and do not contain point 0. If $t > 0$ and $\beta_i(\{t\}) > 0$, then applying the second inequality in (21) to functions g_j for $j \neq i$, we obtain that for such t there is an interval such that $[t, t+\varepsilon) \cap M_j = \emptyset$ for all $j \neq i$. Hence $\beta_j([t, t+\varepsilon]) = 0$, and by (9) $h_j = const$ on this interval for each $j \neq i$. Then formula (11) implies that function $g_i(t)$ is proportional to $f_i(t)$ and hence is strictly increasing on $[t, t+\varepsilon)$. This contradicts $t \in M_i$, and hence $\beta_i(\{t\}) = 0$. This completes the proof of Proposition 3.

The continuity of functions g_i and h_i on $[0,1)$ follows from the continuity of β_i on $[0,1)$ Equality $\varphi_i(\beta) = m_i$ follows from Proposition 2, definition of M_i and (10). Thus, part 1 of Lemma 1 is proved. Part 2 of Lemma 1 was proved in Proposition 2.

Proof of Lemma 2

Given an E-point $(\beta_i, i \in U)$ denote $a_0 = \min_i \inf\{t : t \in M_i\}$. Using $g_i(0) = 0$, $m_i > 0$, continuity of g_i and the definition of m_i, we obtain that $a_0 > 0$. It is obvious also that $a_0 < 1$. It proves point (a) of Lemma 2.

If $t \notin M_i$ for all $i \neq i_0$, then by the continuity of functions g_i, there is an interval (a,b) such that $t \in (a,b)$ and $(a,b) \cap M_i = \emptyset$ for all $i \neq i_0$. Then, by point (b) of Lemma 1, $\beta_i'(t) = 0$ for $i \neq i_0$, and by (9), $h_i(t) = const$ on (a,b) for all

$i \neq i_0$. Hence by (11) $g_{i_0}(t)$ is proportional to $f_{i_0}(t)$ and function $g_{i_0}(t)$ is strictly increasing on this interval, and hence $t \notin M_{i_0}$. Denote $c = \min_i \inf\{a : (a,b) \cap M_i = \emptyset\}$. Then $c \in M_i$ for some $i \in U$ and, by the same argument, g_i is strictly increasing on (c,b). But this is impossible by definition of M_i. This contradiction proves point (b) of Lemma 2.

Let us prove point (c) of Lemma 2. If $(t, t+\varepsilon) \subseteq M_i$ for all $i \in U$, then we are done. In the opposite case there is $i_1 \in U$ and $t_1 \in (t, t+\varepsilon)$ such that $t_1 \notin M_{i_1}$. Continuity of g_{i_1} implies that there is an interval (a_1, b_1) such that $t_1 \in (a_1, b_1) \subseteq (t, t+\varepsilon)$ and $(a_1, b_1) \cap M_{i_1} = \emptyset$. If $(a_1, b_1) \subseteq M_i$ for all $i \neq i_1$, then (a_1, b_1) and $U \setminus \{i_1\}$ satisfy the properties of statement (c). Otherwise there is $i_2 \in U$ and $t_2 \in (a_1, b_1)$ such that $t_2 \notin M_{i_2}$, and so on. By point (b) of Lemma 2 we obtain, in no more than $(m-2)$ steps, an interval and a set J, which satisfy properties of point (c). This competes the proof of Lemma 2.

Proof of Lemma 3

If components of $\beta(\cdot)$ are continuous on $(0,b)$ then equality (11) holds on $(0,b)$ and can be rewritten in the form

$$\frac{g_i(t)}{g_i(a)} = \frac{f_i(t)}{f_i(a)} \frac{h_i(a)}{h_i(t)} \prod_{j \in U} \frac{h_j(t)}{h_j(a)}. \tag{25}$$

Multiplying (25) over all $i \in J = J(a,b)$, and using (14) we get

$$1 = \left(\prod_{i \in J} \frac{f_i(t)}{f_i(a)} \right) \left(\prod_{i \in J} \frac{h_i(a)}{h_i(t)} \right) \left(\prod_{j \in J} \frac{h_j(t)}{h_j(a)} \right)^{|J|} \quad \text{for } t \in (a,b).$$

This is equivalent to

$$\prod_{j \in J} \frac{h_j(t)}{h_j(a)} = \left(\prod_{i \in J} \frac{f_i(t)}{f_i(a)} \right)^{-1/(|J|-1)} = \frac{f_i(a)}{f_i(t)} \frac{r_i(t|J)}{r_i(a|J)} \quad \text{for } t \in (a,b), \tag{26}$$

where the last equality follows from definition (1) of $r_i(t|J)$. Substituting (26) and the right equality from (14) into (25), we obtain an equality which together with (14) implies the left equalities in (15) and (16). The right equality in (15) follows from the left equality and (9). Formulas (3) and (15) imply that $\beta_i' > 0$ for $i \in J$, and the right equality in (14) and (9) imply that $\beta_i' = 0$ for $i \notin J$. This proves point 1 of Lemma 3.

If the right equalities in (15) and (16) hold on (a,b) then the left equalities in (15) and the right equalities in (14) hold on $[a,b)$ due to (9). Substituting these equalities into (25) and using definition (1) of $r_i(t|J)$, we obtain the left equalities in (14). This proves point 2 of Lemma 3.

Let us prove point 3 of Lemma 3. By point (a) of Lemma 1, $\beta(\cdot)$ for an E-point is continuous for any interval, and therefore, by (9) and (11), $h_i(t)$ and $g_i(t)$ are continuous for $i \in U$. If $(a,b) \subseteq M_i$ for all $i \in J$ and $(a,b) \bigcap M_i = \emptyset$ for all $i \notin J$ then, by definition of m_i, $g_i(t) = m_i$ for $i \in J,\ t \in [a,b)$. Point (b) of Lemma 1 implies that $\beta_i'(t) = 0$ for $i \notin J,\ t \in [a,b)$, and hence $h_i(t) = h_i(a)$ on $[a,b)$. So, formula (14) holds on $[a,b)$. This proves sufficiency.

Let (a,b) be a proper interval and $J = J(a,b)$ be a set of active players for an E-point. Then by point 1 of Lemma 3, (15) and (16) hold. It follows from (16) and (3) that functions $g_i(t)$ decrease on (a,b) for nonactive players and, hence, $(a,b) \bigcap M_i = \emptyset$ for all $i \notin J$. It follows from (16) that $\beta_i'(t) > 0$ on (a,b) for active players and, by point (b) of Lemma 1, $(a,b) \subseteq M_i$ for all $i \in J$. This completes the proof of necessity and thus the proof of Lemma 3.

Proof of Lemma 4

First we prove Propositions 4 and 5.

Proposition 4 *Let (a,b), $b < 1$, be a proper interval for an E-point, and $J = J(a,b)$ be a set of active players. Then there is $c > b$ such that $(a,c) \cap M_i = \emptyset$ for all $i \notin J(a,b)$.*

Proof Consider any $i \notin J(a,b)$. Formulas (16) for $g_i(t)$ and (3) imply that function g_i is strictly decreasing on (a,b). Hence $b \notin M_i$. The continuity of g_i implies our statement.

Proposition 5 *Let (a,b) be a proper interval for an E-point, J be a set of active players, and $a_0 \leq a < b < 1$. If $i \notin J$, then $(a,1] \cap M_i = \emptyset$.*

In other words, if player i is inactive on (a,b) then he is inactive forever.

Proof Let us call an interval (a,b) a *maximal* (proper) interval if there is no proper interval (a,c) with $b < c$. By the definition of a proper interval it means that for any $\varepsilon > 0$ there is a point $t \in (b, b+\varepsilon)$ such that either $t \notin M_k$ for some $k \in J$ or $t \in M_j$ for some $j \notin J$. Proposition 4 shows that only the first alternative is possible.

Let $\left(a,b'\right)$ be a maximal proper interval containing (a,b). The definition of a proper interval and a maximal proper interval imply that $J\left(a,b'\right) = J(a,b) = J$. Proposition 4 adds that $g_i\left(b'\right) < m_i$ for all $i \notin J$. If $b' = 1$ then Proposition 5 is proved. If $b' < 1$ then by Proposition 4 there is $c > b'$ such that $(a,c) \cap M_i = \emptyset$ for all $i \notin J$, i.e. there are no new players on (a,c), and in particular $g_i(t) < m_i$ for all $t \in (a,c)$. The definition of a maximal interval implies, as we explained above (the first alternative), that there is a point $t \in \left(b',c\right)$ such that $t \notin M_k$ for some $k \in J$. The continuity of function g_k at t implies that $(t, t+\varepsilon) \cap M_k = \emptyset$ for some $\varepsilon > 0, t+\varepsilon < c$. By point (c) of Lemma 2 and point 3 of Lemma 3 in the interval $(t, t+\varepsilon)$ there is a proper interval (a_1, b_1) . From $(a,c) \cap M_i = \emptyset$ for all $i \notin J$ and $(t, t+\varepsilon) \cap M_k = \emptyset$ for some $k \in J$ we obtain that $J\left(a_1, b_1\right) \subset J$ and this is a strict inclusion. If $b_1 = 1$, and hence $c = 1$ we done because $g_i(t) < m_i$ for all $t \in (a, c)$. If not, we can apply the same construction to the proper interval (a_1, b_1) but with smaller number of active players. In no more than $(m-2)$ steps we will obtain $b_k = 1$, otherwise we will have a contradiction with point (b) of Lemma 2.

Using Propositions 4 and 5, we can prove now that E-point has a structure described by vector $\mathbf{a} \in \mathcal{A}$. Let $t \notin M_i, a_0 \leq t < 1$. The continuity of g_i implies that $(t, t+\varepsilon) \cap M_i = \emptyset$ for some $\varepsilon > 0$. By point (c) of Lemma 2 and point 3 of Lemma 3 there is a proper interval $(a, b) \subseteq (t, t+\varepsilon)$. By the definition of a proper interval we have $i \notin J(a, b)$. Hence by Proposition 5, $(a, 1) \cap M_i = \emptyset$. Since $a_0 < t + \varepsilon$ and ε is arbitrary small, we have that $(t, 1) \cap M_i = \emptyset$. This implies immediately that $a_0 \in M_i$, and $M_i = (a_0, a_i)$ for some number a_i. By (b) of Lemma 2 there are at least two i such that $a_i = 1$. So, we obtain that an E-point is defined by a vector $\mathbf{a} \in \mathcal{A}$. Applying one by one the first equality in (15) to (b_0, b_1), (b_1, b_2) and so on, we get (4) and expression for $h_{i,k}$. This completes the proof of Lemma 4.

Proof of Lemma 5

Before proving Lemma 5 we consider a new mapping, $\mathbf{a} \to \beta(\cdot|\mathbf{a})$ for an arbitrary $\mathbf{a} \in \mathcal{A}_1$. For such $\mathbf{a}$ let us define densities β_i' by equalities $\beta_i' = 0$ for $t \notin (a_0, a_i)$ and by (4) for $t \in (a_0, a_i)$, $i \in U$, where $\mathbf{b} = \mathbf{b}(\mathbf{a}) = (b_0, b_1, \cdots, b_s)$, $s = s(\mathbf{a})$, $h_{i,k}$, $J_k = J_k(\mathbf{a})$, $k = 0, \cdots, s-1$, are defined just as for the case $\mathbf{a} \in \mathcal{A}$. In this case corresponding measures $\beta_i(\cdot|\mathbf{a})$ may not necessary satisfy (12). We consider also $h_i(t|\mathbf{a})$ and $g_i(t|\mathbf{a})$ defined by (9) and (11). These formulas imply that $h_i(t|\mathbf{a})$ and $g_i(t|\mathbf{a})$ are continuous. With such definition the right equalities in (15) and (16) are valid for any interval $\left(b_k, b_{k+1}\right)$ with $J = J_k$. The same reasoning as in the proof of Lemma 3 shows that (14) and the left equalities in (15) and (16) are also valid for this case.

Proposition 6 *Let* $\mathbf{a}, \mathbf{a}' \in \mathcal{A}_1$, *and* $\mathbf{a}$ *differs from* $\mathbf{a}'$ *only by one coordinate. If* $a_j' < a_j$ *then* $\beta_i\left(x|\mathbf{a}'\right) > \beta_i(x|\mathbf{a})$ *for any* $i \in U$ *such that* $i \neq j$, $a_i > a_j'$, *and any* x, $a_j' < x \leq a_i$.

Proof Formula (9) and continuity of h_i imply that β_i and h_i satisfy

$$\beta_i(x|\mathbf{a}) = -\int_{a_0}^{x} \frac{h_i'(t|\mathbf{a})}{f_i(t)} dt = \frac{1}{f_i(a_0)} - \frac{h_i(x|\mathbf{a})}{f_i(x)} - \int_{a_0}^{x} \frac{h_i(t|\mathbf{a}) f_i'(t)}{f_i^2(t)} dt, \tag{27}$$

and hence it suffices to prove that $h_i\left(t|\mathbf{a}'\right) < h_i(t|\mathbf{a})$ for $a_j' < t \leq a_i$.

It follows from (2) and (3) that for any J with $|J| > 2$ and any $i, j \in J$

$$\frac{r_i(t|J \setminus \{j\})}{r_i(c|J \setminus \{j\})} = \frac{r_i(t|J)}{r_i(c|J)} \left(\frac{r_j(t|J)}{r_j(c|J)}\right)^{1/|J|-2} < \frac{r_i(t|J)}{r_i(c|J)} \quad \text{for } 0 < c < t \leq 1. \tag{28}$$

Let $l = \min[k : \ b_k > a'_j]$, where b_k are the coordinates of vector $\mathbf{b}(\mathbf{a})$. First, using (28) we will prove that

$$\frac{h_i\left(t|\mathbf{a}'\right)}{h_i(t|\mathbf{a})} \leq \frac{h_i\left(b_l|\mathbf{a}'\right)}{h_i\left(b_l|\mathbf{a}\right)} \ \text{ for } b_l < t \leq a_i. \tag{29}$$

After that we will prove that $\left(h_i(t|\mathbf{a}')/h_i(t|\mathbf{a})\right) < 1$ for $a'_j < t \leq b_l$. Together with (29) this will imply that $h_i\left(t|\mathbf{a}'\right) < h_i(t|\mathbf{a})$ for $a'_j < t \leq a_i$

Let (c, d) be a proper interval of the type $\left(b_k, b_{k+1}\right)$ with $k \geq l$ and let J be a set of active players on this interval corresponding to $\mathbf{a}$. Obviously (c, d) is also a proper interval corresponding to $\mathbf{a}'$ and the set of corresponding active players is $J \setminus \{j\}$ in case $c < a_j$, or J in case $c \geq a_j$. In both cases the left equality in (15) and (28) imply that $\left(h_i\left(t|\mathbf{a}'\right)/h_i(t|\mathbf{a})\right) \leq \left(h_i\left(c|\mathbf{a}'\right)/h_i(c|\mathbf{a})\right)$ for $c < t \leq d$. Applying this inequality sequentially to the intervals $\left(b_r, b_{r+1}\right)$, $r = k, k-1, \ldots, l$, we get (29).

Let us consider now the interval $\left(a'_j, b_l\right)$. It is obvious that $h_i\left(a'_j|\mathbf{a}'\right) = h_i\left(a'_j|\mathbf{a}\right)$. In case $l > 0$ we can apply the above arguments to the interval $\left(a'_j, b_l\right)$, so that $h_i\left(t|\mathbf{a}'\right) < h_i(t|\mathbf{a})$ in this case. In case $l = 0$ we have $h_i(t|\mathbf{a}) = 1$, and according to the left equality in (15) $h_i\left(t|\mathbf{a}'\right) = r_i(t)/r_i\left(a'_0\right) < 1$ for $a'_0 < t \leq b_l$. This completes the proof of Proposition 6.

To prove point (a) of Lemma 5 let us consider functions $z_i(x|b) := \beta_i(x|(b, x, \cdots, x))$, for $0 < b < x \leq 1, \ i \in U$.

Proposition 7 *There exist* $\nu_i(x)$ *such that if* $b < \nu_i(x)$ *then* $z_i(x|b) > 1$.

Proof Proposition 6 imply that $z_i(x|b)$ decreases when b increases. The definition of $\beta_i(x|\mathbf{a})$ implies that $z_i(x|b) = -\int_b^x \frac{r'_i(t)}{f_i(t)} dt \Big/ r_i(b)$. Using l'Hospital rule we get

$$\lim_{b \to 0} z_i(x|b) = -\lim_{b \to 0} \left(\frac{d}{db} \int_b^x \frac{r'_i(t)}{f_i(t)} dt \Big/ \frac{d}{db} r_i(b) \right) = \lim_{b \to 0} \frac{1}{f_i(b)} = +\infty.$$

This competes the proof of Proposition 7.

Let us consider $\beta_i(\cdot|\mathbf{a})$ for $\mathbf{a} = \mathbf{a}(b), 0 < b < 1$, where vector $\mathbf{a}(b)$ is defined by the basic construction. Then $\beta_i(\cdot|\mathbf{a}(b))$ coincides with continuous part of $\beta_i(\cdot|b), i \in U$. Using this fact we are ready to prove Lemma 5.

Proposition 7 implies that $s_i \geq \nu_i(1)$. Indeed,

$$\beta_i(1_-|b) \geq \beta_i\left(a_i(b)_-|b\right) = \beta_i\left(a_i(b)|\mathbf{a}(b)\right) \geq z_i\left(a_i(b)|b\right), \tag{30}$$

where the last inequality follows from Proposition 6 and the fact that vector $(b, a_i(b), \cdots, a_i(b))$ can be obtained from vector $\mathbf{a}(b)$ by moving to the right into $a_i(b)$ all nonzero coordinates of $\mathbf{a}(b)$ which are less than $a_i(b)$, and moving to the left into $a_i(b)$ all coordinates of $\mathbf{a}(b)$ which are greater than $a_i(b)$. The last operation does not change the value of $\beta_i\left(a_i(b)|\mathbf{a}(b)\right)$. If $s_i < \nu_i(1)$ then by Proposition 7 the righthand side of (30) is greater than 1 for $s_i < b < \nu_i(1)$ and the lefthand side is less or equal 1 by the basic construction. This contradiction completes the proof of point (a) of Lemma 5.

It remains to prove points (b) – (d). For this aim we prove next proposition.

Proposition 8 *Let* $b' < b$, *and* $\mathbf{a} = \mathbf{a}(b)$, $\mathbf{a}' = \mathbf{a}\left(b'\right)$, $\beta_i\left(1_-|b\right)$, $\beta_i\left(1_-|b'\right)$, $i \in U$, *are from the basic construction. If either* $a_i < 1$ *or* $a_i = 1$ *and* $\beta_i\left(1_-|b\right) = 1$ *then* $a_i' < a_i$. *If* $a_i = 1$ *and* $\beta_i\left(1_-|b\right) < 1$ *then* $\beta_i\left(1_-|b'\right) > \beta_i\left(1_-|b\right)$.

Proof We will consider sequentially i from $V_1, \ldots, V_s$, where $V_k = V_k(b)$, $1 \le k \le s = s(b)$, are the sets of players from the basic construction. Recall that only one player v may be not in the union of V_k. The case of $i = v$ will be considered at the end of the proof.

Let $i \in V_1(b)$. Then $\beta_i(1_-|b) = \beta_i\left(a_i|\mathbf{a}\right) = \beta_i\left(a_i|(b, a_i, \ldots, a_i)\right)$. The last equality holds by definition of $\beta_i\left(a_i|\mathbf{a}\right)$. Suppose that $a_i' \ge a_i$. In this case $\beta_i\left(a_{i-}|b'\right) = \beta_i\left(a_i|\mathbf{a}'\right) = \beta_i\left(a_i|\widetilde{\mathbf{a}}'\right)$, where vector $\widetilde{\mathbf{a}}'$ is obtained from vector $\mathbf{a}'$ by reducing to a_i the values of the coordinates which are greater than a_i. Vector $\widetilde{\mathbf{a}}'$ can be obtained also from vector $(b, a_i, \ldots, a_i)$ by sequentially moving the first coordinate from b to $b' < b$, and may be some of coordinates from a_i to $a_j' < a_i$. Then, according to Proposition 6 $\beta_i\left(a_{i-}|b'\right) > \beta_i\left(a_{i-}|b\right)$. If $\beta_i\left(a_{i-}|b\right) < 1$, then, according to the basic construction, $a_i = 1$ and either $a_i' < 1$ or $a_i' = 1$ and $\beta_i\left(1_-|b\right) < \beta_i\left(1_-|b'\right) \le 1$. It proves Proposition 8 for this case. If $\beta_i\left(a_{i-}|b\right) = 1$, then inequality $\beta_i\left(a_{i-}|b'\right) > \beta_i\left(a_{i-}|b\right)$ contradicts the fact that $\beta_i\left(1|b'\right) = 1$. So, our supposition that $a_i' \ge a_i$ is wrong. It completes the proof of Proposition 8 for $i \in V_1(b)$.

Suppose we proved Proposition 8 for all i from $V_1(b), \ldots, V_{l-1}(b)$ for some l, $2 \le l \le s(b)$. Let $i \in V_l(s)$. Then $\beta_i(1_-|b) = \beta_i\left(a_i|\widetilde{\mathbf{a}}\right)$, where vector $\widetilde{\mathbf{a}}$ is obtained from the vector $\mathbf{a}$ by reducing to a_i the values of coordinates from J_l. Suppose that $a_i' \ge a_i$. In this case $\beta_i\left(a_{i-}|b'\right) = \beta_i\left(a_i|\mathbf{a}'\right) = \beta_i\left(a_i|\widetilde{\mathbf{a}}'\right)$, where vector $\widetilde{\mathbf{a}}'$ is obtained from vector $\mathbf{a}'$ by reducing to a_i the values of coordinates which are greater than a_i. Vector $\widetilde{\mathbf{a}}'$ can be obtained from vector $\widetilde{\mathbf{a}}$ by sequentially moving the first coordinate from b to $b' < b$, all coordinates from $V_1(b), \cdots, V_{l-1}(b)$ to the left, and maybe some other coordinates from a_i to $a_j' < a_i$. The same arguments as for the case V_1 imply Proposition 8 for all i from $V_l(b)$, and by induction for all $i \in \bigcup_{k=1}^{s} V_k$.

It remains to consider the case when $i = v \notin \bigcup_{k=1}^{s} V_k$. In this case $a_v = b_s < 1$, $a_j \le 1$ and$\beta_j(a_j) = 1$ for all $j \ne v$. Since $\mathbf{a}', \mathbf{a} \in \mathcal{A}_1$ and we have proved that $a_j' < a_j$ for all $j \ne v$, we obtain that $a_v' \le \max_{j \ne v} a_j' < \max_{j \ne v} a_j = b_s$. This completes the proof of Proposition 8.

Continuity of $a_i(b)$ and $\beta_i(1_-|b)$ follows from the construction of $\mathbf{a}(b)$ and $\beta(\cdot|b)$. Remaining statements of points (b) and (c) of Lemma 5 follow from Proposition 8 and definition (18) of s_i. The first part of point (d) follows from definition of b_* and point (a). The same properties and the fact that $\mathbf{a}(b) \in \mathcal{A}$ imply the second part of point (d). This completes the proof of Lemma 5.

Basic construction for Example 3

The basic construction, applied to Example 3, imply that $\beta(\cdot|b)$ is given by (7). Thus, it remains to find $b_k = b_k(b)$, $J_k = J_k(b)$ for $1 \leq k \leq s$ and $s = s(b)$. Let $b = b_0$ be small enough so that $\overline{a}(b) < 1$. Denote

$$u_0 = 1, u_k = u_k(\mathbf{a}) = \prod_{l=0}^{k-1} \left(\frac{f(b_l)}{f(b_{l+1})} \right)^{\frac{|U_l|}{|U_l|-1}} = h_k \prod_{l=0}^{k-1} \frac{f(b_l)}{f(b_{l+1})}$$

$$= h_k \frac{f(b_0)}{f(b_k)} \quad \text{for } k = 1, \ldots, s-1.$$

Integrating (7), we get

$$\int_{b_k}^{b_{k+1}} \beta_i'(t)dt = \frac{u_k - u_{k+1}}{d_i |J_k| f(b_0)} \text{ for such } i \text{ that } a_i \geq b_{k+1},\ 0 \leq k \leq s-1. \tag{31}$$

Since $v_1 \leq v_j$ for $j > 1$, it follows from (7) with $k = 0$ and from the definition of $\mathbf{a}(b)$ that $\beta_i(b_1) = 1$ for $i \in V_1$ and $\beta_i(b_1) = v_1/v_l < 1$ for $i \in U_l$, $l > 1$. Therefore $J_1 = J_1(\mathbf{a}) = U_1$. Similarly, for $\mathbf{a} = \mathbf{a}(b)$ we have $J_k = J_k(\mathbf{a}) = U_k$ for $2 \leq k \leq s-1$, $s(\mathbf{a}) = n-1$ if $|V_n| = 1$, $s(\mathbf{a}) = n$ if $|V_n| > 1$, and

$$\beta_i(b_k) = 1,\ a_i = b_k \text{ for } i \in V_k; \quad \beta_i(b_k) = \frac{v_k}{v_l} < 1 \quad \text{for } i \in V_l,\ l > k. \tag{32}$$

From these relations we can find $b_k = b_k(b_0)$. Let $v_0 = 0$. Formulas (31) and (32) imply that for $\mathbf{a}(b)$ we have

$$1 - \frac{v_k}{v_{k+1}} = \frac{u_k - u_{k+1}}{v_{k+1}|U_k| f(b_0)} \quad \text{for } 0 \leq k \leq s-1.$$

It follows from here that

$$u_{k+1} = u_k \left(\frac{f(b_k)}{f\left(b_{k+1}\right)} \right)^{(|U_k|)/(|U_k|-1)} = 1 - e_{k+1} f(b_0) \quad \text{for } \ 0 \leq k \leq s-1,$$

and hence

$$\frac{f\left(b_{l+1}\right)}{f(b_l)} = \left(\frac{1 - e_l f(b_0)}{1 - e_{l+1} f(b_0)} \right)^{(|U_l|-1)/(|U_l|)} \quad \text{for } 0 \leq l \leq s-1,$$

where $e_0 = 0$. Multiplying this equalities over l from 0 to k, we obtain that $b_k = b_k(b)$ are given by (8) for $0 < b \leq p_s = b_*$.

Denote $p_0 = 1$. If $p_k \leq b < p_{k-1}$, $1 \leq k \leq s(\mathbf{a})$ then the above calculations show that $a_i(b) = 1$ for $i \in U_{k-1}$ and b_l are given by (8) for $l \leq k-1$. This completes the basic construction for Example 3.

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