The Special Seven, 1823

Call a positive real number *special* if it has a decimal representation that consists entirely of digits 0 and 7. For example, $\frac{700}{99} = 7.\overline{07} = 7.070707\ldots$ and 77.007 are special numbers. What is the smallest n such that 1 can be written as a sum of n special numbers?

Solution: Suppose $1 = x_1 + x_2 + \cdots + x_n$ where $x_1, x_2, \ldots, x_n$ are special and $n \leq 9$. For $k = 1, 2, 3, \ldots$, let a_k be the number of elements of $\{x_1, x_2, \ldots, x_n\}$ whose k^{th} decimal digit is 7. Then

$$1 = \frac{7a_1}{10} + \frac{7a_2}{10^2} + \frac{7a_3}{10^3} + \cdots,$$

which yields

$$\frac{1}{7} = 0.\overline{142857} = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots.$$

Hence $a_1 = 1, a_2 = 4, a_3 = 2, a_4 = 8,$ etc. In particular, this implies that $n \geq 8$. On the other hand,

$$x_1 = 0.\overline{700},\ x_2 = x_3 = 0.\overline{07},\ x_4 = x_5 = 0.\overline{077777},\ \text{and}\ x_6 = x_7 = x_8 = 0.\overline{000777}$$

are 8 special numbers whose sum is

$$\frac{700700 + 2(70707) + 2(77777) + 3(777)}{999999} = 1.$$

Thus the smallest n is 8.