An Odd End, 1869

Think of a whole number. If you multiply together the ten numbers consisting of your number and the next nine consecutive numbers, you get a large number that ends in some zeros. For example, if your choice is 11, then the answer you get is the product $11 \cdot 12 \cdot 13 \cdot \dots \cdot 20 = 670442572800$. Notice that the last digit before the zeros is even, and more often that not this will be the case. There are some starting numbers, however, which will lead to the last non-zero digit of that long product being odd. **What is the lowest such choice?**

Solution: The product of the ten numbers is

$$n \cdot (n+1) \cdot (n+2) \cdot \dots \cdot (n+9).$$

Consider all the prime factors of that product. For the last non-zero digit to be odd, all the factors of 2 must be taken care of in the zeros: the number of factors of 2 must be less than or equal to the number of factors of 5. Two of the ten numbers will have at least one factor of 5, with at most one of them having a factor of more than one 5. So if 5^y, say, is in the list of ten, the number of factors of 5 is $y+1$. So we need $x \leq y+1$ But the ten numbers include five evens, at least two of which are multiples four, and at least one of which is a multiple of eight. So the minimum possible value of x is $1+1+1+2+3=8$. So the minimum value of y is 7. Thus we'll look for a list of ten numbers that include $5^7 = 78125$, starting at the lowest possible.
$78116 - -78125$ gives eight 5's but more than eight 2's;
$78117 - -78126$ gives eight 5's eight 2's, as required.
So the lowest starting number is 78117.

Solution: Originally, I asked 'what is the rightmost non-zero digit of that smallest product?' I submitted this solution: For each positive integer n, let

$$a_n = \frac{(n+9)!}{(n-1)!}$$

Let k denote the smallest positive integer for which the rightmost nonzero digit of a_k is odd. Factor a_n as a product of prime powers:

$$a_n = n(n+1)(n+2)\cdots(n+9) = 2^{e_1}3^{e_2}5^{e_3}\cdots.$$

Among the ten factors $n, n+1, \dots, n+9$, five are even and their product can be written $2^5 m(m+1)(m+2)(m+3)(m+4)$. If m is even then $m(m+2)(m+4)$ is divisible by 16 and thus $e_1 \geq 9$. If m is odd, then $e_1 \geq 8$. If $e_1 > e_3$, then the rightmost nonzero digit of a_n is even. If $e_1 \leq e_3$, then the rightmost nonzero digit of

a_n is odd. Hence we seek the smallest n for which $e_3 \geq e_1$. Among the ten numbers $n, n+1, \ldots, n+9$, two are divisible by 5 and at most one of these is divisible by 25. Hence $e_3 \geq 8$ if and only if one of $n, n+1, \ldots, n+9$ is divisible by 5^7. The smallest n for which a_n satisfies $e_3 \geq 8$ is thus $n = 5^7 - 9$, but in this case the product of the five even numbers among $n, n+1, \ldots, n+9$ is $2^5 m(m+1)(m+2)(m+3)(m+4)$ where m is even, namely $(5^7 - 9)/2 = 39058$. As noted earlier, this gives $e_1 \geq 9$. For $n = 5^7 - 8 = 78117$, the product of the five even numbers among $n, n+1, \ldots, n+9$ is $2^5 m(m+1)(m+2)(m+3)(m+4)$ with $m = 39059$. Note that in this case $e_1 = 8$. Indeed, $39059 + 1$ is divisible by 4 but not by 8, and $39059 + 3$ is divisible by 2 but not by 4. Compute the rightmost nonzero digit as follows. The odd numbers among $n, n+1, \ldots, n+9$ are $7811\underline{7}, 7811\underline{9}, 7812\underline{1}, 7812\underline{3}, 78125 = 5^7$ and the product of the even numbers $78118, 78120, 78122, 78124, 78126$ is $2^5 \cdot 39059 \cdot 39060 \cdot 39061 \cdot 39062 \cdot 39063 = 2^5 \cdot 3905\underline{9} \cdot (2^2 \cdot 5 \cdot 195\underline{3}) \cdot 3906\underline{1} \cdot (2 \cdot 1953\underline{1}) \cdot 3906\underline{3}$. (For convenience, we have underlined the needed unit digits.) Having written $n(n+1)\cdots(n+9)$ as $2^8 5^8$ times a product of odd factors not divisible by 5, we determine the rightmost nonzero digit by multiplying the units digits of these factors. It follows that, for $n = 5^7 - 8$, the rightmost nonzero digit of a_n is the units digit of $7 \cdot 9 \cdot 1 \cdot 3 \cdot 9 \cdot 3 \cdot 1 \cdot 1 \cdot 3 = (9 \cdot 9) \cdot (7 \cdot 3) \cdot (3 \cdot 3)$, namely 9.