Math 1165 **April 12, 2007** **Test 2**

Your name ______________________________

There are 158 points available on this test. You must **show all your work**.

1. (12 points) Solve the decanting problem for containers of sizes 399 and 379; that is find integers x and y satisfying

$$399x + 379y = d$$

where d is the GCD of 399 and 379.

2. (12 points) Find the base 7 representation of each of the numbers below.

   (a) 2007

   (b) 13.2

   (c) 1/3

   (d) Find positive integers a, b, c, and d such that $343 \cdot a + 49 \cdot b + 7 \cdot c + d = 2345$.

3. (8 points) Find the base -4 representation of each of the numbers below.

   (a) 2007

   (b) 13.5

4. (10 points) Let Z denote the integers. Let R be the relation on Z defined by $xRy \Leftrightarrow y \equiv x (\text{mod } 9)$. In other words, $xRy \Leftrightarrow 9 \mid (y - x)$.

   (a) Prove that R an equivalence relation on Z.

   (b) Prove that if $a \equiv b$ (mod 9) and $c \equiv d$ (mod 9), then $a \cdot c \equiv b \cdot d$ (mod 9).

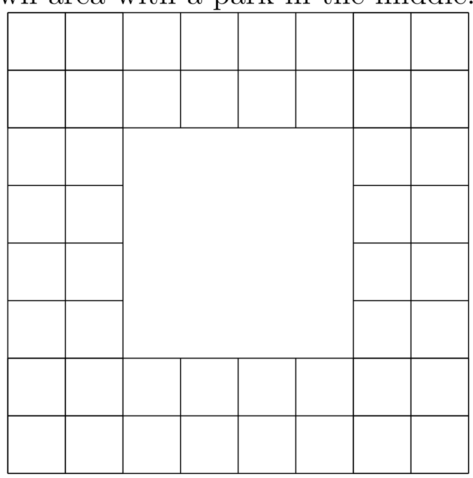
5. (12 points) Recall that the Fibonacci numbers are defined by $F_0 = 1, F_1 = 1$ and for all $n > 1$, $F_n = F_{n-2} + F_{n-1}$.

   (a) Find the units digit of the Fibonacci number F_{2007}.

   (b) Find the congruence class (the cell) of F_{2007} modulo 9. In other words, what is the remainder when the Fibonacci number F_{2007} is divided by 9?

6. (8 points) Solve the equation $1d2d_9 = 306d_7$ for the digit d. For credit, you must show your work.

7. (20 points) Consider the 'annular' grid of unit squares. You can think of this as a design for a downtown area with a park in the middle.



   (a) How many square subregions (of all sizes) does the figure have?

   (b) How many rectangles have sides determined by the grid lines above?

   (c) How many paths of length 16 are there from the lower left corner to the upper right corner?

8. (20 points) Let $X = \{1, 2, 3, 4, 5\}$. Let R be the relation on X defined by $xRy \Leftrightarrow y = x + 2 \vee y = x - 2$.

   (a) Is R a transitive relation? Prove your answer.

   (b) Is R an antisymmetric relation? Prove your answer.

   (c) Let $\overline{R}$ denote the compliment of R. Is $\overline{R}$ reflexive. Prove your answer.

   (d) Draw the digraph of $R \circ R$. Is $R \circ R$ an equivalence relation. If not, which properties does it lack? If so, what partition does it induce?

   (e) Is $R \cup (R \circ R)$ an equivalence relation. If not, which properties does it lack? If so, what partition does it induce?

9. (16 points) Let $X = \{1, 2, 3, 4\}$. Each relation on X is a 4×4 Boolean matrix (we assume the natural ordering on X). On all but the first part, it is enough to draw a digraph.

   (a) How many relations are there on X?

   (b) Find an equivalence relation on X that has exactly eight ordered pairs.

   (c) Find a relation on X that

      i. is symmetric, not antisymmetric and not transitive and

      ii. has exactly eight ordered pairs.

   (d) Find a relation on X that

      i. is reflexive, not symmetric and not transitive and

      ii. has exactly eight ordered pairs.

10. (20 points) Let $S = \{1, 2, 3, 4, 5, 6\}$.

    (a) How many four-element subsets does S have?

    (b) How many four-digit numbers can be made using the members of S as digits if repetition of digits is allowed.

    (c) How many four-digit numbers can be made using the members of S as digits if repetition of digits is not allowed.

    (d) How many four-digit multiples of 45 can be made using the members of S as digits if repetition of digits is allowed.

    (e) How many four-digit multiples of 18 can be made using the members of S as digits if repetition of digits is not allowed.

11. (12 points) We are building 'words' of length 5 using the letters a, b, c, d and e.

    (a) How many words use all five letters?

    (b) How many words use all five letters and have the subword abc ?

    (c) How many of the words in part (a) have the a appearing before the e?

    (d) How many five letter words can be built if letters can be reused?

12. (8 points) Consider the six-digit number $N = \underline{abcdef}$, where a, b, c, d, e, f are digits. Suppose the six-digit number $\underline{bcdefa}$ obtained by moving the leftmost digit a of N to the right end is exactly $3N$. Let x denote the five-digit number $\underline{bcdef}$.

    (a) Write a relation that relates x and a.

    (b) Find two six-digit numbers N that satisfy the property above.