March 27, 2000

Your name ______________________________

It is important that you **show your work**. The total value of this test is 108 points.

1. (16 points) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.

   a. How many non-empty subsets does S have?

   Solution: $2^7 - 1 = 127$.

   b. How many non-empty subsets of S consist entirely of odd numbers?

   Solution: $2^4 - 1 = 15$.

   c. How many subsets of S have exactly 4 elements?

   Solution: $C_4^7 = \binom{7}{4} = 35$.

   d. How many four element subsets of S contain exactly two odd numbers?

   Solution: $C_2^4 \cdot C_2^3 = 6 \cdot 3 = 18$.

2. (8 points) Suppose A, B, and C are finite sets of numbers such that $A \cap B = \phi$, $B \cap C = \phi, |A| = 13, |B| = 15, |C| = 12, |A \cap C| = 2$. Furthermore, $|A \cup B| = 28$. Find each of the following:

   (a) $|A \cap \overline{C}|$

   Solution: Draw a venn diagram or use the inclusion-exclusion principle to find $|A \cap \overline{C}| = 11$.

   (b) $|A \cup B \cup C|$

   Solution: Again use a diagram or the PIE to get $|A \cup B \cup C| = 38$

3. (15 points) Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7\}$, and $C = \{1, 5, 8\}$. Recall that $\times$ denotes Cartesian product and $\overline{X}$ denotes the complement of X with respect to $\mathcal{U}$. Find each of the following. Recall that $A \oplus B$ denotes the symmetric difference of A and B.

   (a) $|\,(A \cap \overline{B}) \times (\overline{A} \cap C)|$

   Solution: $(A \cap \overline{B})$ has only 3 elements and $\overline{A} \cap C)$ is the singleton $\{8\}$, so the cartesian product set has 3 ordered pairs.

   (b) $|\,(A \oplus B) \cup C\,|$

   Solution: $|\,(A \oplus B) \cup C\,| = |\{1, 2, 3, 5, 6, 7, 8\}| = 7$.

   (c) $|\,(A \times A) \cup (A \times B) \cup (A \times C)|$

   Solution: Think about this one. Its the set of all ordered pairs whose first member is in A and whose second member is in $A \cup B \cup C$. So count $A \cup B \cup C$ to find 8 members. Therefore $|\,(A \times A) \cup (A \times B) \cup (A \times C)| = 5 \cdot 8 = 40$.

4. (12 points) Let A and B be sets. Let f be a function from A to B. What is meant by

   (a) f is a one-to-one function.

   Solution: f is one-to-one if for each element b of B there is at most one element a of A such that $f(a) = b$. Equivalently, if $a \neq a'$ then $f(a) \neq f(a')$.

   (b) f is an onto function.

   Solution: For every b in B there is at least one a in A such that $f(a) = b$. Equivalently, every b in B is used up.

   (c) f is a bijective function.

   Solution: Just a function that is both 1-1 and onto.

5. (12 points) Reproduce the proof that $[0,1] \approx [0,1] \times [0,1]$.

   Solution: Let $x = 0.x_1x_2\ldots$ and define $f : [0,1] \longrightarrow [0,1] \times [0,1]$ by

   $$f(x) = (0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots).$$

   We must prove that f is both one-to-one and onto. To see that f is one-to-one, take two different numbers x and y in $[0,1]$, and find the first decimal place where they differ. If $x_i = y_i$ for $i = 1 \ldots k$ and $x_k \neq y_k$, the $f(x) \neq f(y)$ because they have different first coordinates if k is odd and different second coordinates if k is even. To see that f is onto, let $(u,v) \in [0,1] \times [0,1]$. Suppose $u = 0.u_1u_2u_3\ldots$ and $v = 0.v_1v_2v_3\ldots$. Then let $w = 0.u_1v_1u_2v_2\ldots$. Notice that $f(w) = (u,v)$. Thus f is both one-to-one and onto, hence f is a bijection.

6. (15 points) Define what is meant by the characteristic function of a set A, where $\mathcal{U}$ denotes the universal set. Recall that for any sets A and B, $f_{A\cup B} = f_A + f_B - f_Af_B$ and $f_{A\cap B} = f_A \cdot f_B$. Use characteristic functions and the two identities to prove that $\cap$ distributes over $\cup$; that is, for any sets A, B, and C,

   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

   Solution: $f_{A\cap(B\cup C)} = f_A \cdot f_{B\cup C} = f_A(f_B + f_C - f_B \cdot f_C)$ which, in turn, is just $f_Af_B + f_Af_C - f_Af_Bf_C$, whereas $f_{(A\cap B)\cup(A\cap C)}$ is exactly the same function.

7. (20 points) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

   (a) How many four-digit numbers can be made using the digits of S, assuming repetition of digits is not allowed?

   Solution: $P_4^8 = 8!/4! = 1680$.

   (b) How many four-digit numbers can be made using the digits of S, assuming repetition of digits is allowed?

   Solution: $E_4^8 = 8^4 = 4096$.

   (c) How many four-digit numbers can be made using the digits of S, assuming the digits must be in increasing order from left to right? For example 1345 counts, but 1156 does not.

   Solution: $C_4^8 = 70$.

   (d) How many even four-digit numbers can be made using the digits of S, assuming repetition of digits is not allowed?

   Solution: The units digit must be one of the four even digits, and once that is selected, we can choose the other digits in $7 \cdot 6 \cdot 5$ ways. Thus there are 840 such numbers.

8. (10 points) Give either an algbraic or a combinatorial proof of the identity

$$C_r^n + C_{r+1}^n = C_{r+1}^{n+1}.$$

Solution: See the lecture notes for the algebraic proof. The combinatorial proof depends on taking an $n+1$-element set S whose $n+1^{\text{st}}$ element, lets call it z is distinguished from the rest. There are C_r^n ways to choose an $r+1$-element subset of S that include z, and C_{r+1}^n ways that don't include z thus accounting for all $r+1$-element subsets of S.