Math 1165 **Test 1**

September 25, 2000 Your name ____________________________

1. (10 points) Find the base 6 representation of 2000.

   Solution: 13132_6

2. (10 points) Find the base 6 representation of 1/5.

   Solution: $0.\overline{1}_6$

3. (10 points) Find a pair of relatively prime integers m and n for which $\frac{m}{n} = 1.2\overline{3}$. Two numbers are relatively prime if their greatest common divisor is 1.

   Solution: $m = 37$ and $n = 30$.

4. (10 points) Find a base 7 digit d such that $2d16_8 = d405_7$.

   Solution: $2d16_8 = 1024 + 64d + 8 + 6 = 1038 + 64d$. On the other hand, $2d16_8 = d405_7 = 343d + 196 + 5 = 343d + 201$. Therefore, $343d - 64d = 837$, which holds when $d = 3$.

5. (10 points) Find the best (winning) move in the game of Bouton's Nim $(17, 13, 12, 11)$.

   Solution: take 7 from the pile with 17, leaving the position $(10, 13, 12, 11)$.

6. (12 points) Let $M = 161,161$ and let $N = 12,376$.

   (a) Compute $LCM(M, N)$
   Solution: LCM$= 2^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23$.

   (b) Compute $GCD(M, N)$
   Solution: GCD$= 7 \cdot 13 = 91$.

   (c) Find the number of divisors of M.

   Solution: $M = 7^2 \cdot 11 \cdot 13 \cdot 23$ and $N = 2^3 \cdot 7 \cdot 13 \cdot 17$ so 1. LCM$= 2^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, GCD$= 7 \cdot 13$, and M has $(2+1)(1+1)(1+1)(1+1) = 24$ divisors.

7. (20 points) Look at the four equations below.

$$\begin{array}{lcl} 2 & = & 2 \cdot 1 \\ 2+4 & = & 3 \cdot 2 \\ 2+4+6 & = & 4 \cdot 3 \\ 2+4+6+8 & = & 5 \cdot 4 \end{array}$$

a. Write the next three equations in the sequence.

Solution:

$$\begin{array}{lcl} 2+4+6+8+10 & = & 6 \cdot 5 \\ 2+4+6+8+10+12 & = & 7 \cdot 6 \\ 2+4+6+8+10+12+14 & = & 8 \cdot 7 \end{array}$$

b. If the four equations above correspond to $k = 1, 2, 3,$ and 4, what is the nth equation?

Solution:

$$2+4+6+8\ldots+2n \quad = \quad (n+1) \cdot n$$

c. Prove by mathematical induction that the nth equation is true for all integers $n \geq 1$.

Solution: The base case: $2 = (2-1) \cdot 2$. Assume $P(n) : 2+4+6+8\ldots+2n = (n+1) \cdot n$. To prove $P(n+1) : 2+4+6+8\cdots+2n+2(n+1) = (n+2) \cdot (n+1)$, start with the left side and replace the sum of the first n terms with the right side of $P(n)$. Thus $2+4+6+8\cdots+2n+2(n+1) = (2+4+\cdots 2n)+2(n+1) = (n+1) \cdot n + 2(n+1) = (n+1)(n+2)$, which is the right side of $P(n+1)$. By mathematical induction, it follows that $P(n)$ is true for all $n \geq 1$.

8. (10 points) Find the representations of the integers 1 through 13 in base -6.

Solution: $1, 2, 3, 4, 5, 150, 151, 152, 153, 154, 155, 140, 141$.

9. (15 points) Solve the equation $123x + 456y = 3$ for integers x and y.

   Solution: Use the Euclidean Algorithm to find that $x = -63$ and $y = 17$.

10. (13 points) Prove that $2^n \leq n!$ for all integers $n \geq 4$.

    Solution: First note that $2^4 = 16 < 4! = 24$, so the base case holds. Next assume that $P(n) : 2^n \leq n!$. Then $P(n+1)$ is the statement $2^{n+1} \leq (n+1)!$, where $n \geq 4$. But $2^{n+1} = 2 \cdot 2^n \leq 2 \cdot n! < (n+1) \cdot n! = (n+1)!$, because $n \geq 4$.