February 14, 2000

Your name ______________________________

It is important that you **show your work**. The total value of this test is 110 points.

1. (10 points) Find the base -6 representation of 29.
   $\boxed{125^{-6} = 1 \cdot (-6)^2 + 2 \cdot (-6) + 5 \cdot 1 = 36 - 12 + 5 = 29.}$

2. (20 points)

   (a) Use the division algorithm to find the unique integers r and q satisfying
   $$297 = 73q + r \text{ and } 0 \leq r < 73.$$
   $\boxed{q = 4 \text{ and } r = 5.}$

   (b) Solve the decanting problem for containers of sizes 73 and 297; that is find integers x and y satisfying
   $$73x + 297y = d$$
   where d is the GCD of 73 and 297.
   $\boxed{1 = 118 \cdot 73 - 29 \cdot 297, \text{ so } x = 118,\ y = -29}$

3. (10 points) Find the base 6 representation of each of the following:

   (a) 247 $\boxed{= 1051_6}$

   (b) $8\frac{13}{36}$ $\boxed{= 12.21_6}$

   (c) 0.15 $\boxed{= 0.05\overline{2}}$

4. (15 points)

   (a) Construct the base 6 addition table and the base 6 multiplication table.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	10
2	2	3	4	5	10	11
3	3	4	5	10	11	12
4	4	5	10	11	12	13
5	5	10	11	12	13	14

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10	13	20	23
4	0	4	12	20	24	32
5	0	5	14	23	32	41

   (b) Use the tables in (a) to carry out the multiplication $1051_6 \times 204_6$.
   $\boxed{222524_6 = 18772}$

   (c) Convert the three numbers 1051_6, 204_6, and your answer in (b) to their decimal equivalents and carry out the multiplication in decimal representation to check your answer to (b).
   $\boxed{1051_6 = 247,\ 204_6 = 76,\ \text{and } 247 \times 76 = 18772}$

5. (20 points) Notice that

$$\begin{aligned} 2 &= 2 = 2 \cdot 1 && (1) \\ 2+4 &= 6 = 3 \cdot 2 && (2) \\ 2+4+6 &= 12 = 4 \cdot 3 && (3) \\ 2+4+6+8 &= 20 = 5 \cdot 4 && (4) \end{aligned}$$

   (a) List the next three equations suggested by the pattern.

$$\begin{aligned} 2+4+6+8+10 &= 30 = 6 \cdot 5 && (5) \\ 2+4+6+8+10+12 &= 42 = 7 \cdot 6 && (6) \\ 2+4+6+8+10+12+14 &= 56 = 8 \cdot 7 && (7) \end{aligned}$$

   (b) Given that the four equations above are the $1^{st}, 2^{nd}, 3^{rd}$, and 4^{th}, write the n^{th} equation of the sequence.
   $\boxed{2+4+6+8+10+12+\ldots+2n = (n+1)(n)}$

(c) Use mathematical induction to prove that the n^{th} equation is true for all positive integer values of n.

Solution. The base case is the first equation above. Assume $P(n)$: $2+4+6+8+10+12+\ldots+2n = (n+1)(n)$. To prove $P(n+1)$, note that $2+4+6+8+10+12+\ldots+2n+2(n+1) = (n+1)(n)+2(n+1) = (n+1)(n+2)$. Therefore, by mathematicl induction, $P(n)$ is true for all n.

6. (15 points) Divisors and Prime factorization

   (a) Find two different numbers a and b both of which are multiples of 6 such that each one has exactly 8 positive integer divisors.

   Solution. There are lots of numbers with exactly 8 positive integer divisors. The easiest to work with are $24 = 2^3 \cdot 3, 30 = 2 \cdot 3 \cdot 5, 42 = 2 \cdot 3 \cdot 7$, and $54 = 2 \cdot 3^3$.

   (b) Find the greatest common divisor (GCD) of your a and b.

   All those pairs above have 6 as the GCD.

   (c) Find the least common multiple (LCM) of your a and b.

   Just use the formula $LCM(a, b) = a \cdot b \div GCD(a, b)$.

7. (20 points) Compute the remainders when each n below is divided by the given d.

   (a) $n = 3^{2001}$, $d = 10$

   Solution. Note that $3^2 \equiv -1(\mod 10)$, from which it follows that $3^{2001} \equiv 3^{2\cdot1000} \cdot 3 \equiv (-1)^{1000} \cdot 3 \equiv 3 \mod 10$.

   (b) $n = 5^{2001}$, $d = 7$ $\boxed{R = 6}$

   (c) $n = 123{,}456{,}789{,}012{,}345{,}678$, and $d = 9$ $\boxed{R = 0}$

   (d) $n = 123{,}456{,}789{,}012{,}345{,}678$, and $d = 11$ $\boxed{R = 10}$