

## Lecture 9: Properties of Relations, Equivalence Relations and Partially Ordered Sets

There are four very important properties that binary relations might satisfy. We've seen three of them earlier. They are reflexivity, symmetry, antisymmetry, and transitivity. The definitions of these properties are listed below.

**Definition** Let  $S$  be a set and let  $R$  be a relation on  $S$ . We say that  $R$  is

1. **Reflexive** if for every element  $x \in S$ ,  $xRx$ .
2. **Symmetric** if for every pair of elements  $x, y \in S$ ,  $xRy$  implies  $yRx$ . The notation here allows  $x = y$ .
3. **Antisymmetric** if for every pair of elements  $x, y \in S$ ,  $xRy$  and  $yRx$  implies  $x = y$ .
4. **Transitive** if for every triplet of elements  $x, y, z \in S$ ,  $xRy$  and  $yRz$  implies  $xRz$ .

It is helpful to translate each of these properties into the matrix and digraph models. Reflexivity is especially easy. In the matrix of the relation we simply look at the main diagonal. If the diagonal consists entirely of 1's, the relation is reflexive. If there are any 0's on the diagonal, the relation is not reflexive. In the digraph, we check to see that each vertex has a loop. To test symmetry in the matrix model, look at each pair of symmetrically placed entries,  $a_{ij}$  and  $a_{ji}$ . If each such pair is either both 0's or both 1's, then the relation is symmetric. In the digraph model, a relation is symmetric if each directed edge has an oppositely directed counterpart. The property of antisymmetry is also easy to see in the matrix model. Here we note that in order not to be antisymmetric a relation on  $S$  must have a pair of elements each related to the other. In other words, there must be two elements  $a$  and  $b$  of  $S$  such that  $aRb$  and  $bRa$ . We can call this an *instance* of non-antisymmetry. Since the relation must have the property for all pairs, one instance of non-antisymmetry is enough to keep  $R$  from being antisymmetric. Next we deal with the property transitivity. The matrix model is not very helpful here, but the digraph model is quite helpful. To understand transitivity well requires that we be able to say what it is not. The implication

$$xRy \text{ and } yRz \text{ implies } xRz$$

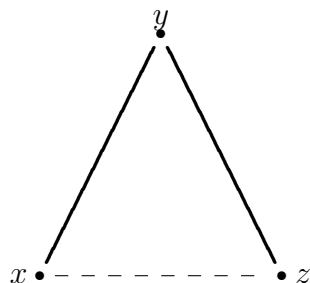
can be written symbolically as

$$xRy \wedge yRz \Rightarrow xRz.$$

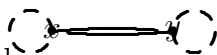
Our goal is to see how the implication can fail. We complete the *truth table* below by noting that there are eight combinations of values T and F that the three  $uRv$  objects can take on, so our table consists of eight rows. In the second array, we complete the second column, the and  $\wedge$  column. Then finally in the third array, we complete the column under the implication  $\Rightarrow$ . Notice that only one F appears in the pertinent column. This owes to the fact that any implication of the form  $F \Rightarrow P$  is true independent of the truth value of  $P$ .

$(xRy \wedge yRz) \Rightarrow xRz$			$(xRy \wedge yRz) \Rightarrow xRz$			$(xRy \wedge yRz) \Rightarrow xRz$				
T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	T	T	F
T	F	T	T	F	F	T	T	F	F	T
T	F	F	T	F	F	F	T	F	F	T
F	T	T	F	F	T	T	F	F	T	T
F	T	F	F	F	T	F	F	F	T	T
F	F	T	F	F	F	T	F	F	F	T
F	F	F	F	F	F	F	F	F	F	T

The next step in understanding the transitivity property is to construct the two instances of *non-transitivity*. The two cases are  $x \neq z$  and  $x = z$  (if either  $x = y$  or if  $y = z$ , then we cannot make the implication false. Since the second row contains the only false implication, and in that case, both  $xRy$  and  $yRz$  are satisfied, we look for a part of the digraph that looks like an *incomplete triangle*.

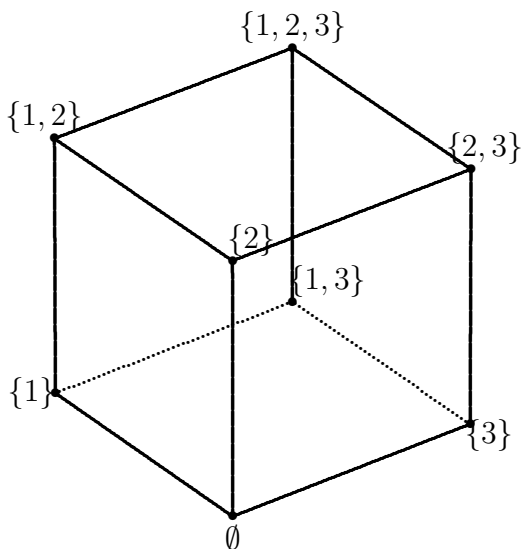


The other picture is what we might call an incomplete dumbbell. In this case  $x = z$ , so the implication is  $xRy$  and  $yRx$  implies  $xRx$ , but  $xRx$  fails. This takes the form of a pair of oppositely directed edges at the ends of which at least one of the two loops is missing.



Next we consider some examples.

1. Let  $S = \{a, b, c, d\}$ , and let  $R = \{(a, a), (b, a), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, c), (d, d)\}$ . Constructing either the matrix or the directed graph enables us to find which of the four properties are satisfied. The relation is reflexive (R), antisymmetric (A), and transitive (T).
2. Let  $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ , and let  $R$  be defined by  $xRy$  if  $x|y$ . Recall that ' $|$ ' means 'divides'. Again the digraph shows that the relation is again R,A,T. These are the defining properties of a partially ordered set. We'll see later that we can use the *Hasse diagram* instead of the entire directed graph.
3. Let  $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ , and let  $R$  be defined by  $xRy$  if  $3|(x - y)$ . Again the digraph shows that the relation is again reflexive (R), symmetric (S), and transitive (T). This combination of properties is called an *equivalence relation*.
4. Let  $T = \{1, 2, 3\}$  and let  $S$  denote the eight subsets of  $T$ . Thus  $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \dots\}$ . Next let  $R$  be the relation on  $S$  defined by  $ARB$  if  $A \subset B$ . We've seen the Hasse diagram for this relation early in the course, but it was disguised at that point. It's just a cube with  $\emptyset$  as the bottom vertex and  $\{1, 2, 3\}$  at the top.



5. Again let  $T = \{1, 2, 3\}$  and let  $S$  denote the eight subsets of  $T$ . Thus  $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \dots\}$ . Next let  $\cong$  be the relation on  $S$  defined by  $A \cong B$

if  $|A| = |B|$ . In other words,  $A \cong B$  means that  $A$  and  $B$  have the same cardinality. Recall that two sets  $A$  and  $B$  are equivalent (we write  $A \cong B$ ) if there is a one-to-one function from one onto the other. Also, recall that the relation  $\cong$  is reflexive, symmetric, and transitive. This combination of properties is called an *equivalence relation*.

6. Let  $S = \{1, 2, 3, 4, 5, 6, 7\}$ , and let  $R$  be defined by  $xRy$  if  $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$ . This is actually very simple to analyze. Two integers  $a$  and  $b$  have the property  $\lfloor \log_2 a \rfloor = \lfloor \log_2 b \rfloor$  precisely when they are caught between the same two powers of 2. That is,  $2^n \leq x \leq y < 2^{n+1}$ . You'll see just how nice the relation is when you construct the matrix. You'll see that the relation is reflexive (R), symmetric (S), and transitive (T) and that the 1's in the matrix occur in blocks along the diagonal. The *cells* of the equivalence relation partition the set  $S$  into three subsets:  $\{1\}$ ,  $\{2, 3\}$ , and  $\{4, 5, 6, 7\}$ .
7. Our next example will be useful below. Let  $S = \{1, 2, 3\}$  and let  $R = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$ .  $R$  is symmetric but not reflexive, antisymmetric, or transitive.
8. As our last example, let  $S$  denote the set of real numbers and define  $R$  as follows:  $xRy \Leftrightarrow x - y$  is an integer. This is another equivalence relation. We'll investigate the cells later in the lecture.

The cells of a relation. In this section we study the cells of an arbitrary relation. We will see that in case the relation is an equivalence relation, something really special happens. The cells are either disjoint or identical. There is no partial overlap. In our definition of cell, each member of the underlying set  $S$  is the name of a cell. The name may or may not be a member of the cell it names. Formally, the definition is this. Suppose  $R$  is a relation of the set  $S$  and  $x \in S$ . The the cell of  $x$ , denoted  $[x]$  is defined by

$$[x] = \{y \in S \mid xRy\}.$$

Look at example 6 above. Note that  $[1] = \{1\}$ ,  $[2] = \{2, 3\}$ ,  $[3] = \{2, 3\}$ ,  $[4] = \{4, 5, 6, 7\}$ , etc. In fact, there are only three different cells. The cells  $[4]$ ,  $[5]$ ,  $[6]$  and  $[7]$  are identical. Notice also that the three cells  $\{1\}$ ,  $\{2, 3\}$ , and  $\{4, 5, 6, 7\}$  are pairwise disjoint (no two of them have a nonempty intersection). Also, their union  $[1] \cup [2] \cup [4]$  is the entire set  $S$ . Such a family of sets is called a partition. In contrast, note that in example 7, none of the members of  $S$  belong to the cell they name. For example  $[1] = \{2, 3\}$ .

We are finally ready for the Big Theorem.

**Theorem.** Let  $S$  be a set and let  $R$  be an equivalence relation on  $S$ . Then the

family of cells partitions  $S$ . Conversely, if  $A_1, A_2, \dots, A_n$  is a family of subsets of  $S$  that partition  $S$ , we can define a relation  $R'$  on  $S$  as follows:  $xR'y$  if  $x$  and  $y$  both belong to the same  $A_i$ . Then  $R'$  is an equivalence relation, and the cells of  $R'$  are precisely the sets  $A_1, A_2, \dots, A_n$ .

**Proof.** To prove the first part, suppose  $R$  is an equivalence relation. We must show that the cells are pairwise disjoint and that their union is all of  $S$ . Said another way, we must show that each element of  $S$  belongs to precisely one cell. First, each element of  $S$  belongs to at least one cell, namely, the one that it names. To prove that no member of  $S$  belongs to two *different* cells, suppose the element  $s$  belongs to  $[x]$  and to  $[y]$ . It will be enough to show that  $[x] = [y]$ . To this end, take an arbitrary element  $z$  in  $[x]$ . Then  $xRz$  by the definition of  $[x]$ . Because  $R$  is symmetric,  $zRx$ . Since  $s \in [x]$ , it follows that  $xRs$ . Since  $R$  is transitive, and  $zRx$  and  $xRs$ , it follows that  $zRs$ . Again because  $R$  is symmetric,  $sRz$ . But  $s \in [y]$  also, so  $yRs$ . Now putting  $yRs$  and  $sRz$  together, we have  $yRz$ , but this is just another way to say  $z \in [y]$ . This string of statements shows that  $[x] \subset [y]$ . It is similar to show that  $[y] \subset [x]$ . Thus  $[x] = [y]$ , and the first part of the theorem is proved.

To see the second part, suppose  $A_1, A_2, \dots, A_n$  is a family of subsets of  $S$  that partition  $S$  and  $R'$  is defined as above. Then  $R'$  is reflexive because each  $x \in S$  belongs to some set  $A_i$ . Now suppose  $xR'y$ . Then  $x$  and  $y$  belong to the same  $A_i$ . Thus  $y$  and  $x$  belong to the same  $A_i$ . So  $R'$  is symmetric. Next suppose  $xR'y$  and  $yR'z$ . Then there is just one  $A_i$  that contains both  $x$  and  $y$ , and just one  $A_j$  that contains both  $y$  and  $z$ . But because the sets  $A_1, A_2, \dots$  partition  $S$ , each element of  $S$  belongs to exactly one of the  $A$ 's. Hence  $i = j$ . It follows that  $R'$  is transitive. The rest of the theorem is easy to see. The theorem can be depicted as follows: Equivalence relation  $R \Rightarrow$  Partition  $\Rightarrow$  Equivalence Relation  $R' = R$  and Partition  $\{A_1, A_2, \dots\} \Rightarrow$  Equivalence Relation  $R' \Rightarrow$  Partition  $\{B_1, B_2, \dots\} =$  Partition  $\{A_1, A_2, \dots\}$ , where  $\Rightarrow$  means 'gives rise to'.

Next we turn our attention to Partially Ordered sets (posets). A partially ordered set is a set  $S$  that is 'partially ordered' by a relation  $\preceq$ . We use  $\preceq$  because it provides a bit more intuition than  $R$  does about its properties. In order to qualify to be a poset,  $\preceq$  must be reflexive, antisymmetric, and transitive (RAT). A relation  $\preceq$  satisfying RAT generalizes the relation  $\leq$  on the real numbers. The best way to understand a finite poset  $S, \preceq$  is to draw its Hasse Diagram. This skeleton can be obtained from the digraph or from the matrix of the relation. The idea is that since we know the relation is a partial ordering, the loops need not be included in the picture, and neither do we need to include the directed edges that are implied by transitivity. Finally, if we

agree to draw the diagram so that all the edges are directed upwards, we need not include arrows for the direction. We can agree that all edges are upwardly directed because the relation is antisymmetric, which implies that there are no ‘doubly directed’ edges. We’ll demonstrate the process of drawing the Hasse diagram from the matrix with two examples.

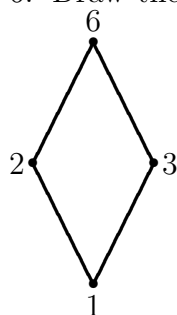
**Example 1** Let  $S = \{1, 2, 3, 6\}$  and let consider the relation divides  $|$ . The matrix is

	1	2	3	6
1	1	1	1	1
2	0	1	0	1
3	0	0	1	1
6	0	0	0	1

Note that the column of the 1 has only one 1 among its four entries. This means that nothing lies below the element 1. So we put 1 at the bottom, and remove the row and the column of 1 to obtain the  $3 \times 3$  matrix

	2	3	6
2	1	0	1
3	0	1	1
6	0	0	1

Now repeat the addition process. You find that the elements 2 and 3 tie as least elements, so we put them at the next level. Then note that 1 is related to both 2 and 3, so we must connect 1 with each of these. Finally remove both the rows and columns of 2 and 3 to get the  $1 \times 1$  matrix for the single element 6. Put the 6 at the top, and note that both 2 and 3 are related to 6. Draw the edges from 2 to 6 and from 3 to 6. Thus the Hasse diagram is



Next we repeat this process for the poset of divisors of 30.

**Example 2** Let  $S = \{1, 2, 3, 5, 6, 10, 15, 30\}$  and let consider the relation divides  $|$ . The matrix is

	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	0	1	0	0	1	1	0	1
3	0	0	1	0	1	0	1	1
5	0	0	0	1	0	0	1	1
6	0	0	0	0	1	0	0	1
10	0	0	0	0	0	1	0	1
15	0	0	0	0	0	0	1	1
30	0	0	0	0	0	0	0	1

This time 1 is again the least element. When the row and column of 1 are eliminated, we get

	2	3	5	6	10	15	30
2	1	0	0	1	1	0	1
3	0	1	0	1	0	1	1
5	0	0	1	0	0	1	1
6	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1
15	0	0	0	0	0	1	1
30	0	0	0	0	0	0	1

The column addition step shows that all three of 2, 3, and 5 are minimal, so we'll put them at the level one up from the 1. Removing the rows and columns of all three of these gives the  $4 \times 4$  matrix

	6	10	15	30
6	1	0	0	1
10	0	1	0	1
15	0	0	1	1
30	0	0	0	1

and we can read from this that the three numbers 6, 10 and 15 are minimal. We put them at the next level and finally put 30. Finally, we connect the least element 1 with those elements at the level one up from the 1 which are related to 1. Thus we have edges from 1 to 2, 1 to 3 and 1 to 5. Next put the three elements 6, 10, and 15 at the level above the 2, 3, 5 level and draw the edges between the pairs represented in the original matrix by the 1's.

