

Lecture 4: Induction and Recursion

In lecture 3, we discussed two important applications of the Mathematical Induction Principle:

(1.) *summation problems*. These are problems that ask for a formula for $F(n) = S_n = a_1 + \dots + a_n$ in terms of $f(n) = a_n$, and

(2.) *divisibility problems*. These are problems in which it is required to show that all numbers of a certain form are divisible by all numbers of a certain type. In this lecture, we continue to develop applications of mathematical induction, beginning with inequalities.

Problem 1. Prove that a geometrical sequence is increasing faster than the sequence of perfect squares. Specifically: prove that for appropriate $n_0 > 0$ and any $n \geq n_0$

$$2^n > n^2.$$

Solution. First of all we have to determine the value of the unknown borderline n_0 . Let's construct the table of values of 2^n and n^2 :

n	1	2	3	4	5	6	...
n^2	1	4	9	16	25	36	...
2^n	2	4	8	16	32	64	...

The table shows that the borderline is probably $n_0 = 5$, i.e., for $n \geq 5$, $2^n > n^2$. Of course this conclusion is based on the particular numerical experiment (incomplete induction). The real proof can be done by mathematical induction.

Step 1. (Basis of induction). Inequality $2^n > n^2$ is true for $n_0 = 5$: $2^5 = 32 > 5^2 = 25$. Of course here (as well as in many similar situations) the initial value of n is not $n_0 = 1$, but depends on the results of our numerical observations.

Step 2. Assume that for some $n \geq n_0 = 5$ we have $2^n > n^2$ (Inductive hypothesis). Using this fact we'll check that $2^{n+1} > (n+1)^2$ and mathematical induction will give us the final result: for any $n \geq 5$

$$2^n > n^2.$$

Let's multiply both sides of the inductive hypothesis inequality by 2, which gives

$$2 \cdot 2^n > 2n^2 \Rightarrow 2^{n+1} > 2n^2.$$

If we can prove that $2n^2 \geq (n+1)^2$ for $n \geq 5$ we'll then have

$$2^{n+1} > 2n^2 \geq (n+1)^2 \Rightarrow 2^{n+1} > (n+1)^2,$$

which would complete the proof. This method is standard: to prove that $A > B$ simply show the two steps: $A > C$ and $C \geq B$ for some appropriate intermediate C .

Let's return to the calculations

$$\begin{aligned} 2n^2 &\geq (n+1)^2 \Leftrightarrow 2n^2 \geq n^2 + 2n + 1 \Leftrightarrow \\ n^2 &\geq 2n + 1 \Leftrightarrow n^2 - 2n \geq 1 \Leftrightarrow n(n-2) \geq 1. \end{aligned}$$

But $n \geq 5$, i.e., $n(n-2) \geq 5 \cdot 3 = 15 > 1$.

Problem 2. Prove that for suitable $n \geq n_0$

$$n! \geq 5 \cdot 3^n.$$

Find the minimal n_0 .

Problem 3. Prove that

$$n > 5 \log_3 n$$

if $n \geq n_0$. Find the minimal n_0 .

The following example not only illustrates the mathematical induction principle, it also contains one of the fundamental and useful inequalities between arithmetic and geometric means. The original classical proof is due to Cauchy.

Definition. Let x_1, \dots, x_n be non-negative numbers. Then the arithmetic mean m_n is given by the formula

$$m_n = \frac{x_1 + x_2 + \dots + x_n}{n}.$$

The geometric mean g_n (of the same numbers) is given by

$$g_n = \sqrt[n]{x_1 \cdots x_n}.$$

Theorem. (AMGM Inequality) For any $n \geq 2$ and all $x_i \geq 0$, $i = 1, 2, \dots, n$

$$m_n = \frac{x_1 + \dots + x_n}{n} \geq g_n = \sqrt[n]{x_1 \cdots x_n}.$$

The inequality is strict, except in the case when all x_i are equal: $x_1 = x_2 = \dots = x_n$.

Proof. Induction, up and down. If $n = 2$ then

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}. \quad (*)$$

In fact

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2} \Leftrightarrow x_1 + x_2 - 2\sqrt{x_1 x_2} \geq 0 \Leftrightarrow$$

$$(\sqrt{x_1})^2 + (\sqrt{x_2})^2 - 2\sqrt{x_1}\sqrt{x_2} = (\sqrt{x_1} - \sqrt{x_2})^2 \geq 0.$$

The last inequality is obvious as well as the fact that the equality (*) holds if and only if $\sqrt{x_1} - \sqrt{x_2} = 0$, i.e., $x_1 = x_2$.

For $n = 4$ we can get the same result using (*) several times

$$\begin{aligned} \frac{x_1 + x_2 + x_3 + x_4}{4} &= \frac{\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}}{2} \geq \\ &\geq \frac{\sqrt{x_1x_2} + \sqrt{x_3x_4}}{2} \geq \sqrt{\sqrt{x_1x_2}\sqrt{x_3x_4}} = \sqrt[4]{x_1x_2x_3x_4}. \end{aligned}$$

We can continue this process to get the inequality for $n = 2, 4, 8, 16, \dots$ i.e. all the integer powers of 2: 2^k , $k \geq 1$. For instance,

$$\begin{aligned} \frac{x_1 + x_2 + x_3 + \dots + x_8}{8} &= \frac{\frac{x_1+x_2+x_3+x_4}{4} + \frac{x_5+x_6+x_7+x_8}{4}}{2} \\ &\geq \frac{\sqrt[4]{x_1x_2x_3x_4} + \sqrt[4]{x_5x_6x_7x_8}}{2} \text{ and so forth.} \end{aligned}$$

(Give the formal proof. It is sufficient to check only the transition from $n = 2^k \rightarrow n = 2^{k+1}$).

This is induction up. Let's see how to apply induction down.

Lemma. If for fixed n

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1x_2 \dots x_n} \quad (**)$$

then the same is true for $n - 1$:

$$\frac{y_1 + y_2 + \dots + y_{n-1}}{n-1} \geq \sqrt[n-1]{y_1 \dots y_{n-1}}.$$

(Again $x_i \geq 0$, $1, 2, \dots, n$; $y_j \geq 0$, $j = 1, 2, \dots, n-1$.)

Proof. Let's substitute in (**)

$$x_1 = y_1, x_2 = y_2, \dots, x_{n-1} = y_{n-1}, x_n = \frac{y_1 + y_2 + \dots + y_{n-1}}{n-1}.$$

Then

$$\begin{aligned} \frac{x_1 + \dots + x_n}{n} &= \frac{(y_1 + \dots + y_{n-1}) \left(1 + \frac{1}{n-1}\right)}{n} = \frac{y_1 + \dots + y_{n-1}}{n-1} \geq \\ &\sqrt[n]{y_1y_2 \dots y_{n-1} \left(\frac{y_1 + \dots + y_{n-1}}{n-1}\right)} \Leftrightarrow \end{aligned}$$

$$\left(\frac{y_1 + y_2 + \cdots + y_{n-1}}{n-1}\right)^n \geq y_1 y_2 \cdots y_{n-1} \left(\frac{y_1 + \cdots + y_{n-1}}{n-1}\right).$$

Now divide both sides by the positive (why can we assume its positive?) number $\left(\frac{y_1 + \cdots + y_{n-1}}{n-1}\right)$ to get

$$\left(\frac{y_1 + \cdots + y_{n-1}}{n-1}\right)^{n-1} \geq y_1 y_2 \cdots y_{n-1}.$$

Finally, take the $(n-1)^{\text{st}}$ root of both sides to get

$$\left(\frac{y_1 + \cdots + y_{n-1}}{n-1}\right) \geq \sqrt[n-1]{y_1 \cdots y_{n-1}}.$$

One can reach any number n (not necessarily a power of 2) in two steps: jump up until level $2^s > n$ and move down: $2^s \rightarrow 2^s - 1 \rightarrow 2^s - 2 \rightarrow \cdots \rightarrow (n+1) \rightarrow n$. Hence we are done.

Let's now take a step backwards and discuss a topic we've seen already. Sequences. There are really three ways to define an infinite sequence of real numbers. The first way is to list the first few elements followed by ellipsis (three dots). Using this method presents a problem for both the author and reader because there is a chance that the reader will infer something different from what the author meant. Another method for defining a sequence is to specify how each term is obtained from previous terms. Often each term can be obtained from just the one before it. This method is called recursion. The third method is to provide a closed form formula, much like functions are defined symbolically in the calculus. We'll give examples below, and see how to translate from one method to another.

1. Consider the sequence $1, 3, 5, 7, \dots$. Are we on safe ground? Yes, I think so, as long as you know I'm not trying to fool you. The sequence is the odd numbers in numerical order. A way to define it recursively is a. $a_1 = 1$ and b. $a_n = a_{n-1} + 2$ for all $n \geq 2$.
The sequence could be defined non-recursively by the formula $a_n = 2n - 1$.
2. Consider the sequence $1, 2, 4, 8, \dots$. The sequence is (nonnegative) integer powers of 2. A way to define it recursively is a. $a_1 = 1$ and b. $a_n = 2a_{n-1}$ for all $n \geq 2$.
The sequence could be defined non-recursively by the formula $a_n = 2^{n-1}$.
3. Consider the sequence $1, 3, 6, 10, \dots$. The sequence is the positive triangular numbers in numerical order. A way to define it recursively is a. $a_1 = 1$ and b. $a_n = a_{n-1} + n$ for all $n \geq 2$.
The sequence could be defined non-recursively by the formula $a_n = n(n+1)/2$.

4. Consider the sequence 1, 2, 6, 24, 120, 720 . . . This is the sequence of factorials! A way to define it recursively is a. $a_0 = 1 = 0!$ and b. $a_n = na_{n-1}$ for all $n \geq 1$.
The sequence could be defined non-recursively by the formula $a_n = n!$, assuming the ! key is part of our repertoire of symbols.
5. Suppose the sequence is defined by $a_n = 3^n + 1$. Find a recursive definition of it. Solution.

Now we introduce a new notion. It often happens that the general term in a sequence or general step in some numerical algorithm is not given by an explicit formula or a rule, depending on the number n of the term or step, but instead depends on (or can be expressed as) several previous terms or steps. In such cases we say that the sequence or algorithm is recursive or is recursively defined.

Example. Find \sqrt{a} with the help of a calculator or computer but without using the square root key itself. The following procedure has a recursive structure. First find x_0 such that $x_0^2 < a < (x_0 + 1)^2$, i.e., a very rough approximation to \sqrt{a} . After that, define a sequence of numbers as follows: $x_1 = \frac{1}{2}(x_0 + \frac{a}{x_0})$ $x_2 = \frac{1}{2}(x_1 + \frac{a}{x_1})$, . . . $x_n = \frac{1}{2}(x_{n-1} + \frac{a}{x_{n-1}})$. Next we prove that $x_n \rightarrow \sqrt{a}$ as $n \rightarrow \infty$, if this limit exists. Let $x = \lim_{n \rightarrow \infty} x_n$.

$$\begin{aligned} x &= \frac{1}{2}\left(x + \frac{a}{x}\right) \Rightarrow x = \frac{x}{2} + \frac{a}{2x} \Rightarrow \\ &\Rightarrow \frac{x}{2} = \frac{a}{2x} \Rightarrow x^2 = a \Rightarrow x = \sqrt{a}; \end{aligned}$$

and the convergence is extremely fast. Numerical illustration: $a = 2$, $x_0 = 1$. Then

$$\begin{aligned} x_1 &= \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} \\ x_2 &= \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12} \\ x_3 &= \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) = \frac{577}{408} = 1.414215686 \end{aligned}$$

On the other hand, a calculator gives $\sqrt{2} = 1.414213562$, i.e. the third approximation gives 5 proper digits. The 4th approximation

$$x_4 = \frac{1}{2} \left(\frac{577}{408} + \frac{816}{577} \right) = \frac{665857}{470832}$$

gives 10 proper digits, which is higher than the accuracy of the calculator itself!

This is the special case of *Newton's Method*, a powerful recursive method for numerical solutions of equations or systems of equations.

Example. Some well-known objects in elementary algebra called arithmetic and geometric progressions, are defined recursively. Progressions are often called sequences.

By definition, the sequence $a_n, n = 1, 2, \dots$ is called an arithmetic progression (or arithmetic sequence) if

$$a_{n+1} = a_n + d,$$

for some constant d .

It is easy to see (or to check by mathematical induction) that

$$a_n = a_1 + d(n - 1). \quad (***)$$

For any progression a_1, a_2, a_3, \dots , let S_n denote the sum of the first n terms of the progression.

Problem 4. Prove that for any arithmetic progression with the first term a_1 and difference d then

1. $S_n = n \cdot a_1 + \frac{d(n-1)n}{2}$, and
2. $S_n = \frac{n}{2}(a_1 + a_n)$ (the sum is the average of the first and last terms).

One can use either mathematical induction or the formula $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ (see lecture 3).

The definition of geometric progression is based on the recursive formula

$$a_{n+1} = a_n \cdot q \quad (q \text{ is the ratio of the progression}).$$

Then, analogous to (***), we can prove that

1. $a_n = a_1 \cdot q^{n-1}$
2. $S_n = a_1 \left(\frac{q^n - 1}{q - 1} \right)$.

Verify these formulas.

The following examples are more complicated.

Example Define the sequence a_n by the recursive formula

$$a_{n+1} = 3a_n - 2a_{n-1}$$

together with the “initial conditions”: $a_1 = 1, a_2 = 3$. Find formula for a_n .

At the first stage of induction (not mathematical induction but rather the “experimental” one), we try to guess a formula: $a_1 = 1, a_2 = 3, a_3 = 3 \cdot 3 - 2 \cdot 1 = 9 - 2 = 7, a_4 = 3 \cdot 7 - 2 \cdot 3 = 21 - 6 = 15$ and $a_5 = 3 \cdot 15 - 2 \cdot 7 = 45 - 14 = 31$. Does it look like

$$a_n = 2^n - 1?$$

Now let's use mathematical induction. Let $P(n)$ denote both (a.) $a_{n-1} = 2^{n-1} - 1$ and (b.) $a_n = 2^n - 1$. The base case is $P(2)$: $a_1 = 2^1 - 1$ and $a_2 = 2^2 - 1$.

Suppose that

$$\begin{aligned} a_{n-1} &= 2^{n-1} - 1 \\ a_n &= 2^n - 1. \end{aligned}$$

Then

$$\begin{aligned} a_{n+1} &= 3a_n - 2a_{n-1} = 3(2^n - 1) - 2(2^{n-1} - 1) = \\ &= 3 \cdot 2^n - 3 - 2 \cdot 2^{n-1} + 2 = 3 \cdot 2^n - 2^n - 1 = \\ &= 2 \cdot 2^n - 1 = 2^{n+1} - 1. \end{aligned}$$

The hypothetical formula $a_n = 2^n - 1$ is proved!

The Characteristic Polynomial Method. The following method will enable us to find a closed form formula for certain recurrence relations. The idea is quite similar to a method used in solving certain differential equations. A recurrence relation $a_n = a \cdot a_{n-1} + b \cdot a_{n-2}$ is called *homogeneous* because the sequence $a_n = 0$ satisfies it. Contrast this with $a_n = a \cdot a_{n-1} + b \cdot a_{n-2} + 1$ which is not satisfied by $a_n = 0$. We call the recurrence *second order* because the two previous terms are required to compute a_n . The method we describe here works for all homogeneous first and second order recurrence relations. It also works for higher order recurrences, but, as you will see, we have to find the zeros of a polynomial to make the method work. Thus we will limit our attention to first and second order recurrences. We demonstrate the method with several examples.

Example 1. Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ together with the initial values $a_0 = 3$ and $a_1 = 2$. In step 1 we assume that there is a sequence of the form $a_n = \lambda^n$ that satisfies the recurrence. If so, then $\lambda^n = 3\lambda^{n-1} - 2\lambda^{n-2}$. We can collect all the terms on one side of the equation, and factor to get:

$$\lambda^n - 3\lambda^{n-1} + 2\lambda^{n-2} = \lambda^{n-2}(\lambda^2 - 3\lambda + 2) = 0.$$

The quadratic part of the equation can be factored. It has two roots, $\lambda = 1$ and $\lambda = 2$. Thus we have two solutions $a_n = 1^n = 1$ and $a_n = 2^n$. Next we note that any *linear combination* of solutions is also a solution. In other words, for any numbers c_1 and c_2 , the sequence $a_n = c_1 + c_2 \cdot 2^n$ satisfies the recurrence relation. Now it happens that there are unique values for c_1 and c_2 for which a_n also satisfies the initial conditions. Replacing n with 0 yields $a_0 = c_1 + c_2 \cdot 2^0$ or $c_1 + c_2 = 3$. Replacing n by 1 yields $a_1 = c_1 + c_2 \cdot 2^1$ or $c_1 + 2c_2 = 2$. We can solve these two linear equations simultaneously by subtracting one from the other. Thus $c_2 = -1$ and $c_1 = 4$. The

unique solution to the recurrence with the initial conditions is $a_n = 4 - 2^n$. We can prove this by mathematical induction.

Example 2. Our next example is more interesting. The famous *Fibonacci sequence* $F_n, n \geq 0$ is defined recursively as follows:

$$F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}.$$

Thus first few members of the sequence are given below:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

We can use the technique above to find an explicit formula for F_n . First find the characteristic equation, $\lambda^{n-2}(\lambda^2 - \lambda - 1) = 0$ which we can solve by the quadratic formula to get $\lambda = \frac{1 \pm \sqrt{1+4}}{2}$. Then it remains to find the right c_1 and c_2 . For convenience, let $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$. Thus, the general solution to the recurrence is $F_n = c_1\lambda_1^n + c_2\lambda_2^n$. Then, using the conditions $F_0 = 0$ and $F_1 = 1$, we find $0 = a_0 = c_1 + c_2$ and $1 = a_1 = c_1\lambda_1 + c_2\lambda_2$. Replacing c_2 with $-c_1$ in the second of these and noting that $\lambda_1 - \lambda_2 = -\sqrt{5}$, it follows that $c_1 = -\frac{1}{\sqrt{5}}$ and $c_2 = \frac{1}{\sqrt{5}}$. Thus

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right], n \geq 0.$$

This odd combination of irrational numbers gives the integer Fibonacci numbers! Use this formula to compute the number F_{100} .

Problem. Prove the formula for F_n . Of course, one has to use mathematical induction starting from two initial values of n : $n = 0, n = 1$. It will be the basis of induction.

At the inductive step one must use the following fact: numbers

$$\lambda_1 = \frac{1 - \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 + \sqrt{5}}{2}$$

The number $\lambda_2 = \frac{1+\sqrt{5}}{2} = 1.618034$ is the famous “golden section” or golden ratio from ancient mathematics. It was believed that arches that had this height to width ratio were the most pleasing to the eye.

The last result (formula for F_n) is a very special case of the following general statement.

Example 3. Let $a_{n+1} = 5a_n - 6a_{n-1}$, $a_0 = 1$, $a_1 = 3$. Find the formula for a_n .

Solution. Let's construct the characteristic polynomial: $\lambda^2 = 5\lambda - 6 \Rightarrow \lambda^2 - 5\lambda + 6 = 0$. Solve this to get $\lambda_1 = 2$, and $\lambda_2 = 3$. Then

$$a_n = c_1 \cdot 2^n + c_2 \cdot 3^n$$

$$\text{and } \begin{aligned} a_0 &= 1 = c_1 + c_2 \\ a_1 &= 3 = 2c_1 + 3c_2 \end{aligned}$$

Solve these simultaneously to get $c_1 = 0$ and $c_2 = 1$. Thus $a_n = 0 \cdot 2^n + 1 \cdot 3^n$. Check this result by induction.

In each of the examples above, the characteristic polynomial has distinct zeros. In example 4 we see how the technique differs when there are roots of multiplicity at least two.

Example 4. Let $a_{n+1} = 6a_n - 9a_{n-1}$, $a_0 = 1$, $a_1 = 4$. Find a closed form formula for a_n . Now the characteristic polynomial is $\lambda^2 - 6\lambda + 9 = 0$, which factors into $(\lambda - 3)^2 = 0$, so the root $\lambda = 3$ is repeated. In this case, the usual method does not work because we would not have a pair of *linearly independent* solutions. But in this case the sequences $a_n = 3^n$ and $a_n = n3^n$ satisfy the recurrence and are linearly independent. Thus we can find a pair of constants c_1 and c_2 such that $a_n = c_1 \cdot 3^n + c_2 \cdot n3^n$ uniquely satisfies both the recurrence and the initial conditions. In fact we get $c_1 = 1$ and $c_2 = 1/3$. In the homework you will be asked to work problems of these types and also to find the recurrence relation that gives rise to certain solutions.