Lecture 3. Mathematical Induction

Induction is a fundamental reasoning process in which general conclusion is based on particular cases. It contrasts with *deduction,* the reasoning process in which conclusion logically follows from *axioms.* Axioms are the simplest "obvious facts" (these facts are the results of the long history of the human observations and experience).

Induction plays a very important role in our knowledge about the world and nature. Practically all modern information (scientific and practical) is based on inductive reasoning. Past experience is used as the basis for generalizations about future experience.

Induction in natural sciences cannot be absolute, because it is based on a very large but *finite* number of observations and experiments. We know that in the process of the evolution of such sciences as physics or biology, all the fundamental laws from time to time have been revised. For example, Newton's gravitational law gave way to Einstein's general relativity theory.

Mathematics is probably the most striking example of a deductive science; nevertheless the inductive hypothesis and inductive experiments are essential for mathematics in exactly the same way as for other natural sciences. The role of induction in mathematics is especially important today, when we have such powerful computing instruments for experiments with the equations and numbers. Several examples of induction in mathematics and evolution of the induction hypotheses follow.

Example 1. Since the medieval centuries mathematicians have sought to find a formula $f(n)$ producing nothing but primes for all $n \geq n_0$. For example, the formula $f(n) = n^2 + n + 41$, $n \geq 0$ gives primes for $n = 0\,(f(0) = 41)$, $n = 1\,(f(1) = 43)$, $n = 2\,(f(2) = 47), \ldots,$ and $n = 39(f(39) = 1601)$. All these values are prime. But for $n = 40$ we have

$$f(40) = 40^2 + 40 + 41 = 40 \cdot 41 + 41 = 41^2,$$

which is a composite number.

Euler proved that no polynomial formula $f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0$ with integer coefficients $a_k, a_{k-1}, \ldots, a_0$ can produce only primes for all $n \geq n_0$.

Example 2. In the 16th century, Italian mathematician Tartaglia claimed that $f(n) = 2^n - 1$ gives primes for all odd n. His claim was based, of course, on the following observations:

$$\begin{array}{lll} n = 3, & 2^3 - 1 = f(3) = 7 & (prime), \\ n = 5, & 2^5 - 1 = f(5) = 31 & (prime), \\ n = 7, & 2^7 - 1 = f(7) = 127 & (prime). \end{array}$$

His inductive analysis was very limited. In fact, the next odd $n = 9$ gives

$$f(9) = 2^9 - 1 = 511 = 7 \cdot 73 \text{ (composite)}.$$

It is not difficult to show that for an arbitrary composite number $n = n_1 \cdot n_2$, $1 < n_1,\ n_2 < n$, the number

$$2^n - 1 = 2^{n_1 n_2} - 1 = \left(2^{n_1}\right)^{n_2} - 1$$

is composite. In fact for all real x we can check that

$$(x^n - 1) = (x - 1)\left(x^{n-1} + x^{n-2} + \cdots + x + 1\right)$$

i.e., for $x = 2^{n_1}$

$$\left(2^{n_1}\right)^{n_2} - 1 = \left(2^{n_1} - 1\right)\left(\left(2^{n_1}\right)^{n_2-1} + \cdots + 2^{n_1} + 1\right).$$

This means that if the number

$$2^n - 1 = f(n)$$

is prime, the exponent n *must* be prime. This observation goes to the French mathematician of the 17th century Mersenne. He claimed first that for all prime P, the number $\left(2^P - 1\right)$ also is prime, but recognized shortly that it is not always true. It is true for $p = 2, 3, 5, 7$ but

$$2^{11} - 1 = 2047 = 23 \cdot 89.$$

However, there are many primes of the form $\left(2^P - 1\right)$. They are known as Mersenne primes. The largest known primes are the Mersenne primes. There are special algorithms to recognize Mersenne primes and there are interesting ways to use them in number theory and algebra. It is not known yet whether there is an infinite number of Mersenne primes.

Example 3 (Moser's problem). The following problem arises in combinatorial geometry. Let S denote the unit circle in the plane and let $A_1, \ldots, A_n$ be points on its boundary. Let's connect each pair of the points A_i, A_j by the chords $[A_i, A_j]$ and assume that all the points of intersection inside the circle are distinct (that is, no three chords intersect at one point). These chords divide the circular region into $N = N(n)$ parts: n segments along the boundary of S and convex polygons (including triangles) inside S. Prove that number $N = N(n)$ does not depend on the locations of the points A_i and find it (for all $n = 1, 2, \ldots$).

We start by gathering experimental data. Let's consider small values of n:

$n = 1$,	no chords	$N(1) = 1$
$n = 2$,	one chord	$N(2) = 2$
$n = 3$,	one triangle and three segments	$N(3) = 4$

$n = 4$,	four segments and four triangles	$N(4) = 5$
$n = 5$,	5 segments, 1 pentagon, 10 triangles $N(5) = 16$.	

Our observations (for $n = 1, 2, \ldots 5$) correspond to the conjecture $N(n) = 2^{n-1}$, $n \geq 1$. This is a typical example of the incomplete induction. Our conjecture was based on the finite number of particular cases.

This conjecture (as many other similar ones) is wrong: for $n = 6$ we have $N(6) = 31$, not 32. See the following picture.

It is possible to prove that the correct formula for N is given by

$$N = N(n) = 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{2}.$$

Example 4. Let $1, 3, 5, \ldots$ be the sequence of positive, odd integers. A formula for the general term a_n, $n = 1, 2, \ldots$ has a form

$$a_n = 2n - 1.$$

Our goal is to find the sum

$$S_n = a_1 + a_2 + \cdots + a_n = 1 + 3 + \cdots + (2n-1).$$

Because $S_1 = a_1 = 1$
$S_2 = a_1 + a_2 = 1 + 3 = 4 = 2^2$
$S_3 = a_1 + a_2 + a_3 = 1 + 3 + 5 = 9 = 3^2$ and
$S_4 = 1 + 3 + 5 + 7 = 16 = 4^2$,it makes sense to guess that

$$S_n = n^2$$

We didn't prove this result mathematically. It was only an inductive hypothesis based on the finite number of "observations": $n = 1, 2, 3, 4$.

Our induction is *incomplete.* How can we check $S_n = n^2$ for all $n = 1, 2, \ldots$?

There exists a special method, so-called *mathematical induction,* to prove the theorems of the form such-and-such is true for all positive integers, n. Sometimes it is useful to formalize this by letting $P(n)$ denote the statement about n. Then the proposition becomes $P(n)$ holds for all $n = 1, 2, \ldots$.

Examples of propositions, depending on the positive integer n, $n = 1, 2, \ldots$ are
$P_1(n)$: $S_n = \underbrace{1 + 3 + \cdots + (2n-1)}_{n \text{ terms}} = n^2$
$P_2(n)$: $S_n = 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots n)^2$

$P_3(n)$: For all non-negative real numbers $x_1, x_2, \ldots x_n$

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq n\sqrt{x_1 x_2 \ldots x_n}$$

$P_4(n) : f(n) = n^3 - n$ is divisible by 5 for $n \geq 1$.

Every proof by mathematical induction contains two steps.

Step 1. (*Basis of induction*) Show that $P(1)$ is true.

Step 2. Show that the *truth of* $P(n)$ for arbitrary $n \geq 1$ *implies the truth of* $P(n+1)$.

The statement, "$P(n)$ is true" is called the *inductive hypothesis.* We have to deduce the truth of $P(n+1)$ from the truth of $P(n)$.

$P(1)$ is true, according to Step 1. But then Step 2 implies that $P(2)$ is true, then Step 2 again implies that $P(3)$ is true, etc.

Sometimes one can use mathematical induction in a slightly different form. The form below is called "strong induction" by some authors.

Step 1. (*Basis of induction*) Show that proposition $P(n)$ is true for initial value of n; say n_0, and $n \geq n_0$.

Step 2. Show that $P(n+1)$ is true if all $P(k)$, $n_0 \leq k \leq n$ are true.

Then the principle of mathematical induction gives the truth of $P(n)$, $n \geq n_0$.

Example 4 (continued). We have to prove the proposition $P(n)$:

$$S_n = \underbrace{1 + 3 + \cdots + (2n-1)}_{n \text{ terms}} = n^2,$$

for all $n \geq 1$. We will use the method of mathematical induction.

Step 1. (*Basis of induction*) For $n = 1$, $S_1 = 1 = 1^2$, i.e. $P(1)$ is true. As usual, the first step is simple.

Step 2. Suppose, that for some $n \geq 1$ our proposition $P(n)$ is true, i.e.

$$S_n = 1 + 3 + \cdots + (2n-1) = n^2 \text{ (inductive hypothesis)}$$

We have to deduce from this fact the truth of $P(n+1)$:

$$S_{n+1} = 1 + 3 + \cdots + (2n-1) + (2n+1) = (n+1)^2.$$

But notice that the right side of the induction hypothesis equation and the right side of the equation $P(n+1)$ differ in just one term, namely $2n+1$. Thus

$$S_{n+1} = S_n + (2n+1) = n^2 + 2n + 1 = (n+1)^2.$$

Example 5. Let $S_n = 1 + 2 + \cdots + n$ denote the sum of the first n terms for the series of natural numbers. Using mathematical induction, prove that

$$S_n = 1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}.$$

As a matter of fact, we can deduce the solution of this problem from Example 4. We have

$$\begin{aligned} 1 + 3 + \cdots + (2n-1) &= n^2 \\ (2\cdot 1 - 1) + (2\cdot 2 - 1) + \ldots (2\cdot n - 1) &= n^2 \\ 2\,(1 + 2 + \cdots + n) - \underbrace{(1 + \cdots + 1)}_{n \text{ times}} &= 2\,(1 + 2 + \cdots + n) - n. \end{aligned}$$

That is

$$\begin{aligned} n^2 &= 2\,(1 + 2 + \cdots + n) - n \Longrightarrow \\ n^2 + n &= 2\,(1 + 2 + \cdots + n) \Longrightarrow 1 + 2 + \cdots + n = \frac{n\,(n+1)}{2} \end{aligned}$$

Example 6. Let us try to find the following sum

$$S_n = 1^3 + 2^3 + \cdots + n^3 \text{ for all } n \geq 1.$$

We start with "numerical experiments" and attempt to guess the answer.
$S_1 = 1^3 = 1 = 1^2$
$S_2 = 1^3 + 2^3 = 9 = 3^2$
$S_3 = 1^3 + 2^3 + 3^3 = 36 = 6^2$
$S_4 = 1^3 + 2^3 + 3^3 + 4^3 = 36 + 64 = 100 = 10^2$
But $1 = 1$, $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$ and we can formulate the hypothesis $P\,(n)$:
for all $n \geq 1$,

$$S_n = 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left[\frac{n\,(n+1)}{2}\right]^2.$$

Let's try to prove this proposition using mathematical induction.
Step 1. (*Basis of induction*) This step is trivial.
Step 2. Suppose, that

$$S_n = 1^3 + \cdots + n^3 = \left[\frac{n\,(n+1)}{2}\right]^2.$$

then $S_{n+1} = 1^3 + \cdots + n^3 + (n+1)^3 = S_n + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left[\frac{n^2}{4} + (n+1)\right] = \frac{(n+1)^2}{4}\left(n^2+4n+4\right) = \frac{(n+1)^2(n+2)^2}{4}$, and we are done. Be sure you understand why.

Examples 4 through 6 represent the typical application of mathematical induction to summation problems. If the sequence a_n, $n \geq 1$ is given by the formula

$$a_n = f(n), \quad n = 1, 2, \ldots$$

and we expect that

$$S_n = a_1 + a_2 + \cdots + a_n = F(n), \quad n = 1, 2, \ldots$$

where $F(n)$ is given by another formula, then one can prove the last fact using the same procedure. The key moment (step 2) is the transition from n to $n+1$ based on the relation

$$S_{n+1} = a_1 + \cdots + a_n + a_{n+1} = S_n + a_{n+1},$$

that is, $F(n+1) = F(n) + a_{n+1}$. To find the appropriate expression for $F(n)$ we have to use completely different ideas. In many cases the formula for $F(n)$ is simply known from the very beginning.

Different kinds of applications of the mathematical induction give the divisibility problems. The following problem is typical.

Example 7. Prove that the integer $f(n) = 4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

Proof by mathematical induction:

Step 1. (*Basis of induction*). We have for $n = 1$

$$f(1) = 4^1 + 15 \cdot 1 - 1 = 18 = 9 \cdot 2.$$

Thus $P(1)$ is true.

Step 2. Suppose, that $f(n) = 4^n + 15n - 1$ is divisible by 9 (inductive hypothesis). We must prove that the last statement implies the divisibility of $f(n+1)$ by 9. But

$$\begin{aligned} f(n+1) &= 4^{n+1} + 15(n+1) - 1 = 4 \cdot 4^n + 15n + 15 - 1 = \\ &= 4 \cdot 4^n + \underbrace{4 \cdot 15n - 3 \cdot 15n} + \underbrace{4 - 4} + 14 = \\ &= 4\left(4^n + 15n - 1\right) - 45n + 18 = 4f(n) - 45n + 18. \end{aligned}$$

The first form is divisible by 9 according to inductive hypothesis, the last two have a form $-45n + 18 = 9(-5n+2)$, that is, they are always divisible by 9. The theorem is proved by the principle of mathematical induction.

The central moment here is the representation of $f\,(n+1)$ in the "recursive" form:

$$f\,(n+1) = \alpha f\,(n) + R\,(n)$$

where $R\,(n)$ is divisible by g identically and $f\,(n)$ is divisible by g due to inductive hypothesis. Similar ideas work in all divisibility problems.

Appendix. Summation of the polynomial sequences. One of the important applications of mathematical induction is the proof of different general formulas for the summation of series. Such formulas are useful in the calculus and numerical analysis. Lecture 3 contained the following formulas:

$$\begin{aligned} S_n &= 1+3+...+(2n-1) = n^2, \\ S_n &= 1+2+...+n = \frac{n(n+1)}{2}, \\ S_n &= 1^3+2^3+...+n^3 = \left[\frac{n(n+1)}{2}\right]^2. \end{aligned}$$

The common feature of these three examples is that the general term a_n in the sum $S_n = a_1+...+a_n$ is a polynomial function of the index n. For the first series it is $f_1(x) = 2x-1$, $a_n = f_1(n) = 2n-1$, for the second one $f_2(x) = x$, $a_n = f_2(n) = n$, at last for the third example, $f_3(x) = x^3$, $f_3(n) = a_n = n^3$.

One can also observe that the sum S_n in all three cases is the polynomial function of n and its degree is equal to degree of the general term plus 1!

Say,

$$\begin{aligned} \frac{n(n+1)}{2} &= \frac{1}{2}n^2 + \frac{1}{2}n \\ \left[\frac{n(n+1)}{2}\right]^2 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2. \end{aligned}$$

We can now formulate (without proof) the following recipe.

Suppose, that $a_n = c_m n^m + c_{m-1} n^{m-1} + ... + c_1 \cdot n + c_0$ is the general term of the sequence $a_1, a_2, \ldots, a_n, \ldots$. This term is a polynomial function of the fixed degree m from the argument (index) n. Then

$$S_n = a_1 + ... + a_n = d_{m+1} \cdot n^{m+1} + d_m \cdot n^m + ... + d_1 \cdot n^m$$

is the polynomial function of the degree $m+1$ from the argument (index) n. To find coefficients $d_{m+1}, ..., d_1,$ it is sufficient to solve the following system of linear equations (for $m+1$ variables $d_{m+1}, ..., d_1$) :

$$\begin{array}{rr} n=1 & S_1 = d_{m+1} + ... + d_1 \\ n=2 & S_2 = d_{m+1}\cdot 2^{m+1} + d_m \cdot 2^m + ... + 2d_1 \\ n=3 & S_3 = d_{m+1}\cdot 3^{m+1} + d_m \cdot 3^m + ... + 3d_1 \\ n=m+1 & S_{m+1} = d_{m+1}(m+1)^{m+1} + ...(m+1)d_1. \end{array}$$

When we find coefficients $d_{m+1}, ..., d_1$ it will be possible to prove the formula for S_n using mathematical induction. We used the hypothesis from the recipe only to "design" the answer, but a *known* answer can be checked by mathematical induction.

Example 9. *Find the sum of the squares for the first n natural numbers, i.e. find the sum*

$$S_n = 1^2 + 2^2 + ... + n^2.$$

Solution. The general term $a_n = n^2$ is a polynomial of degree 2. We can try to find S_n in the form (notation for coefficients are different!)

$$S_n = an^3 + bn^2 + cn.$$

But $S_1 = 1^2 = 1,\ S_2 = 1^2 + 2^2 = 5,\ S_3 = 1^2 + 2^2 + 3^2 = 14$. For the coefficients a, b, c we have the system: $\begin{array}{r} 1 = a+b+c \\ 5 = 8a+4b+2c \\ 14 = 27a+9b+3c \end{array} \Rightarrow \begin{array}{r} 1 = a+b+c \\ 3 = 6a+2b \\ 11 = 24a+6b \end{array}$ (Elimination of c from the second and third equations) $\Rightarrow \begin{array}{r} 1 = a+b+c \\ 3 = 6a+2b \\ 2 = 6a. \end{array}$ (elimination of b from the third equatio

Then,

$$a = \frac{1}{3},\ b = \frac{1}{2},\ c = \frac{1}{6}$$

that is, the "hypothetical" formula for S_n has a form

$$\begin{aligned} S_n &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n}{6}\left[2n^2 + 3n + 1\right] = \\ &= \frac{n}{6}\left[2n^2 + 2n + n + 1\right] = \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Now we'll prove this formula by mathematical induction. The basis step is trivial:

$$1 = S_1 = \frac{1\cdot 2\cdot 3}{6}.$$

Suppose (inductive hypothesis) that

$$S_n = \frac{n(n+1)(2n+1)}{6}.$$

Then

$$\begin{aligned}
S_{n+1} &= S_n + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \\
&= \frac{(n+1)}{6}\left[n(2n+1) + 6(n+1)\right] = \frac{(n+1)}{6}\left[2n^2 + 7n + 6\right] \\
&= \frac{(n+1)}{6}\left[2n^2 + 4n + 3n + 6\right] = \frac{(n+1)(n+2)(2n+3)}{6} = \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.
\end{aligned}$$

Formula for S_{n+1} has exactly the same form as for S_n but with replacement of n by $(n+1)$. Induction step is done, theorem is proved.