

May 5, 2000

Your name _____

It is important that you **show your work**.

1. Find a relation R on the set $\{1, 2, 3, 4\}$ satisfying each set of requirements. You may leave your answer in digraph form, matrix form, or as a set of ordered pairs.

(a) Reflexive, antisymmetric and not transitive.

(b) Transitive, symmetric and not antisymmetric.

2. Prove that for any $a \neq 1$,

$$1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

for $n = 0, 1, 2, \dots$

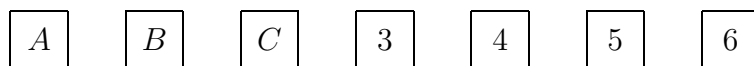
3. Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 317 and 975. In other words, find integers x and y such that $\gcd(317, 975) = 317x + 975y$.

4. Find the remainders when each N is divided by d .

(a) $N = 5^{41}$ and $d = 3$

(b) $N = 123,456,789,101,112$ and $d = 9$

5. Each of the cards shown below has a number on one side and a letter on the other. How many of the cards must be turned over to prove the correctness of each statement below? When this number is not unique, explain why that is the case.



Note: *A* is a vowel and *B* and *C* are not, 3 and 5 are prime numbers and 4 and 6 are not.

- (a) Every card with a vowel on one side has a prime number on the other side.
- (b) Every card has a vowel on one side if and only if it has a prime number on the other side.
- (c) Every card has either a vowel on one side or a prime number on the other side.
- (d) Some card has either a 3 on one side or an *A* on the other.
- (e) Some card has a 3 on one side and an *A* on the other.

6. How many relations R on the set $S = \{1, 2, 3, 4\}$ are there such that

(a) there are no restrictions on R ?

(b) R is both reflexive and antisymmetric?

(c) R satisfies the property that for each $x \in S$, there is exactly one $y \in S$ such that $(x, y) \in R$?

7. (a) Prove that the intersection of two transitive relations on the set A is also transitive.
- (b) Prove that the union of two symmetric relations on the set A is also symmetric.
- (c) Prove that the compliment \overline{R} of a symmetric relation R on the set A is symmetric.
- (d) Give an example that shows that the union of two transitive relations on the set A need not be transitive.

8. Let Z denote the set of all integers. Define R on Z by xRy if $x - y$ is a multiple of 3 (note that 0 is a multiple of 3). Which of the following properties does R satisfy? Give *reasons* for each answer. The reason is roughly four times the value of the correct yes-no answer.

(A) reflexivity

(B) symmetry

(C) transitivity

(D) antisymmetry

If R is an equivalence relation, compute the partition Z/R ; in other words, find the cells of the partition.

10. Find the base 8 representation of each of the following numbers.

(a) 2001

(b) $2^9 + 2^7 + 2^5 + 2^3 + 1$

(c) $3 \cdot 16^3 + 5 \cdot 16 + 11 \cdot 16^{-2}$

(d) Explain how you can find the base 2 representation of a base 8 numeral without converting it into a decimal first.