May 5, 2000

Your name ______________________________

It is important that you **show your work**.

1. Find a relation R on the set $\{1, 2, 3, 4\}$ satisfying each set of requirements. You may leave your answer in digraph form, matrix form, or as a set of ordered pairs.

   (a) Reflexive, antisymmetric and not transitive.
   Solution: There are many examples. One of the simplest is

   $$\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,3)\}$$

   (b) Transitive, symmetric and not antisymmetric.
   Solution: Again, there are many examples. One of the simplest is $\{(1,1),(2,2),(1,2),(2,1)\}$

2. Prove that for any $a \neq 1$,

   $$1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

   for $n = 0, 1, 2, \ldots$.
   Solution: For $n = 0$, we have $1 = \frac{a^1-1}{a-1}$. To show the inductive step, $P(n)$: $1+a+a^2+\cdots+a^n = \frac{a^{n+1}-1}{a-1}$ implies that $P(n+1) : 1+a+a^2+\cdots+a^n+a^{n+1} = \frac{a^{n+2}-1}{a-1}$, start with the left side of $P(n+1)$, and replace the sum f the first $n+1$ terms with the right side of $P(n)$ to get $1+a+a^2+\cdots+a^n+a^{n+1} = \frac{a^{n+1}-1}{a-1}+a^{n+1}$. Find a common denominator and add to get $P(n + 1)$.

3. Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 317 and 975. In other words, find integers x and y such that $gcd(317, 975) = 317x + 975y$.

   Solution: Repeatedly divide, starting with 317 divided into 975. Then unwind the process to get

   $$\begin{aligned} 1 &= 5 - 4 = 5 - (24 - 4 \cdot 5) \\ &= 5 \cdot 5 - 1 \cdot 24 \\ &= 5(317 - 13 \cdot 24) - 1 \cdot 24 \\ &= 5 \cdot 317 - 66 \cdot 24 \\ &= 5 \cdot 317 - 66(975 - 3 \cdot 317) \\ &= 203 \cdot 317 - 66 \cdot 975 \end{aligned}$$

   So we have the solution $x = 203, y = -66$.

4. Find the remainders when each N is divided by d.

   (a) $N = 5^{41}$ and $d = 3$

   Solution: Notice that $5^2 \equiv 25 \equiv 1(\text{mod } 3)$. Therefore $5^{41} = (5^2)^{20} \cdot 5 \equiv 1^{20} \cdot 5 \equiv 5 \equiv 2(\text{mod } 3)$, so the remainder is 2.

   (b) $N = 123{,}456{,}789{,}101{,}112$ and $d = 9$

   Solution: The number is congruent to the sum of its digits, modulo 9. The sum of the digits is 51, which is congruent to 6 modulo 9. Therefore the remainder is 6.

5. Each of the cards shown below has a number on one side and a letter on the other. How many of the cards must be turned over to prove the correctness of each statement below? When this number is not unique, explain why that is the case.

A	B	C	3	4	5	6

   Note: A is a vowel and B and C are not, 3 and 5 are prime numbers and 4 and 6 are not.

   (a) Every card with a vowel on one side has a prime number on the other side.

   Solution: Only three cards need be turned over, the A, the 4 and the 6, to verify this implication.

   (b) Every card has a vowel on one side if and only if it has a prime number on the other side.

   Solution: All seven cards must be turned over since any one of them by itself could falify the statement.

   (c) Every card has either a vowel on one side or a prime number on the other side.

   Solution: The cards with $B, C, 4$, and 6 must be turned over.

   (d) Some card has either a 3 on one side or an A on the other.

   Solution: None.

   (e) Some card has a 3 on one side and an A on the other.

   Solution: Maybe 1, maybe 2, depending on what the first one has on its obverse.

6. How many relations R on the set $S = \{1, 2, 3, 4\}$ are there such that

   (a) there are no restrictions on R?

   Solution: $2^{16} = 65536$

(b) R is both reflexive and antisymmetric?

Solution: $3^6 = 729$. Each symmetric pair $(a,b),(b,a)$ can be filled is with the pairs $(0,0),(0,1),(1,0)$, and their are six such symmetric pairs.

(c) R satisfies the property that for each $x \in S$, there is exactly one $y \in S$ such that $(x,y) \in R$?

Solution: These are just the functions from S to S and there are $4^4 = 256$ of them.

7. (a) Prove that the intersection of two transitive relations on the set A is also transitive.

Solution: Let R and T be transitive relations on the set A. Then $xR \cap Ty \wedge y(R \cap T)z \implies xRy \wedge yRz \wedge xTy \wedge yTz$ which implies that $xRz \wedge xTz$ because both R and T are transitive. Thus $x(R \cap T)z$ which is what we needed to prove.

(b) Prove that the union of two symmetric relations on the set A is also symmetric.

Solution: Again using the notation above, $x(R \cap T)y \implies xRy \vee xTy$ which means that $x(R \cup T)y$. In other words, $R \cup T$ is symmetric.

(c) Prove that the compliment $\overline{R}$ of a symmetric relation R on the set A is symmetric.

Solution: If R is symmetric, then $x\overline{R}y \implies (x,y)$ does not belong to R, which implies that (y,x) does not belong to R, which means that $y\overline{R}x$. Done.

(d) Give an example that shows that the union of two transitive relations on the set A need not be transitive.

Solution: One easy example is $R = \{(1,2)\}$ and $T = \{(2,3)\}$.

8. Let Z denote the set of all integers. Define R on Z by xRy if $x-y$ is a multiple of 3 (note that 0 is a multiple of 3). Which of the following properties does R satisfy? Give *reasons* for each answer. The reason is roughly four times the value of the correct yes-no answer.
(A) reflexivity

Solution: yes
(B) symmetry

Solution: yes
(C) transitivity

Solution: yes
(D) antisymmetry

Solution: no

If R is an equivalence relation, compute the partition Z/R; in other words, find the cells of the partition.

Solution: The cells are given by

$$[0] = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$$

$$[1] = \{1, -2, 4, -5, 7, -8, \ldots\}$$

and

$$[2] = \{2, -1, 5, -4, 8, -7, \ldots\}.$$

9. Five card poker hands. A five-card poker hand is a set of five playing cards selected from a deck of 52 ordinary playing cards (there are four *suits* each with 13 *denominations*.

   (a) How many five-card poker hands are there altogether?
   Solution: $C_5^{52} = 2,598,960$

   (b) How many five-card poker hands consist entirely of hearts?
   Solution: $C_5^{13} = 1287$

   (c) How many five-card poker hands have three hearts and two clubs?
   Solution: $C_3^{13} \cdot C_2^{13} = 78 \cdot 286 = 22308$

   (d) How many five-card poker hands have three cards of one denomination (value) and two of some other denomination? Such hands may be described as a *full house*.
   Solution: $C_1^{13} \cdot C_3^4 \cdot C_1^{12} \cdot C_2^4 = 52 \cdot 72 = 3744$

10. Find the base 8 representation of each of the following numbers.

    (a) 2001
    Solution: Repeated division yields 3721_8

    (b) $2^9 + 2^7 + 2^5 + 2^3 + 1$
    Solution: Convert to binary first to get 1010101001_2 which is easily converted to base 8: 1251_8.

    (c) $3 \cdot 16^3 + 5 \cdot 16 + 11 \cdot 16^{-2}$
    Solution: This takes some arithmetic and eventuall yields 30120.026.

    (d) Explain how you can find the base 2 representation of a base 8 numeral without converting it into a decimal first.
    Solution: Each base 8 numeral give rise to three binary digits. For example, $6_8 = 110_2$, and $3_8 = 11_2$ so $636_8 = 110011110_2$.