# Lecture 7: Combinatorics

If $\mathcal{U}$ is a finite universal set, $|\mathcal{U}|$ = cardinality($\mathcal{U}$) $< \infty$ and $A \subset \mathcal{U}$ then, of course, $|A| < |\mathcal{U}|$. The subject of combinatorics is the counting of the number of elements (cardinality) for different subsets of the different universal sets. Usually, these subsets are related with the combinatorics of objects, such as cards, letters, digits etc. Possible applications: games, probability theory, codes and so forth.

Let's start from the *Addition principle.*

1. Suppose, that $A, B \subset \mathcal{U}$, $|A|, |B| < \infty$ and $A, B$ are disjoint: $A \cdot B = \emptyset$. Then

$$|A \cup B| = |A| + |B|$$

   Proof. Each element $x \in A \cup B$ appears in exactly one of the sets ($A$ or $B$), but not in both together.

2. If $A \cap B \neq \emptyset$, then $A$ and $B$ overlap and

$$|A \cup B| < |A| + |B|$$

   because in the right part we count the elements from $A \cap B$ twice. To get equality we have to subtract $|A \cap B|$ the right side to make up for the double counting of members of $A \cap B$:

$$|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B| - |AB|.$$

The formula above is known as the principle of inclusion-exclusion, which we can abbreviate as PIE. Using PIE for two sets and rules of the sets operations we can prove PIE for more sets.

When there are three sets,

$$\begin{aligned}
&|A \cup B \cup C| = \\
&\quad |(A \cup B) \cup C| = \\
&\quad |A \cup B| + |C| - |(A \cup B)C| = \\
&\quad |A| + |B| - |AB| + |C| - |AC \cup BC| = \\
&\quad |A| + |B| + |C| - |AB| - |BC| - |AC| + |ACBC|.
\end{aligned}$$

But $ACBC = ABCC = ABC$ (because $CC = C$).

Thus,

$$|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |BC| - |AC| + |ABC|$$

If the sets $A, B, C$ are disjoint, i.e.

$$AB = \emptyset,\ BC = \emptyset,\ AC = \emptyset$$

then the formula is much simpler:

$$|A \cup B \cup C| = |A| + |B| + |C|$$

The notation $A \oplus B$ is used to denote the *symmetric difference* of the sets $A$ and $B$: $A \oplus B = A \cup B - A \cap B$. Let's mention also a few other formulas, which are consequences of the addition principle:

$$\begin{array}{rcl} |\overline{A}| & = & |\mathcal{U}| - |A|, \\ |A - B| & = & |A| - |AB| \\ |A \oplus B| & = & |A| + |B| - 2|AB|. \end{array}$$

*Multiplication principle*

The following statement is simple but fundamental. It's known as the *product rule* or *multiple choice lemma.*

*Lemma.* Suppose that we have multi-step (K-steps) experiment. The first step can result in $n_1$ different outcomes (opportunities), the second can result in $n_2$ different outcomes (for each outcome at the first step)—the last one (the Kth step) can result in $n_k$ outcomes. The total number of outcomes for compound K-step experiments is equal given by

$$N = n_1 n_2 \cdots n_k.$$

If the opportunities at the first stage are denoted by $A_1, A_2, ..., A_{n_1}$, and those for the second step by $B_1, B_2, ..., B_{n_2}$, etc. and those for the last stage by $X_1, X_2, ..., X_{n_K}$ then every outcome of the compound experiment is represented by a "word" of the form

$$A_{i_1} B_{i_2} \ldots X_{i_K}$$

The proof of the lemma is simple. We can use either the graph representation of our experiment (or words), or the coordinate method. The idea of the coordinate method is clear for $K = 2$. Word of the length $K = 2$, $A_{i_1} B_{i_2}$ can be represented as a cell on the "checker board" type rectangular table (Fig 1)I hope to get this filled in summer 2001.

The total number of cells is equal to the area of rectangle, i.e. $n_1 \cdot n_2$ in agreement with lemma.

The number of different words is the number of parts on the graph, i.e.

$$N = n_1 \cdot n_2 \cdots n_K \cdot$$

To finish the proof we can apply the principle of mathematical induction.

*Applications of the lemma*

1. What is the number of ways to arrange $n$ distinct objects (in one line)? Such arrangements are known as *permutations* (or full permutations). The total number of permutations of $n$ objects is denoted by $P_n$. We'll prove that

$$P_n = n(n-1)\cdots 2\cdot 1 = n!$$

This number $n(n-1)\cdots 2\cdot 1 = n!$ is called n-factorial. To prove the formula, we can use multiple choice process, described in the lemma above. The first object in permutation can be selected in $n$ ways. For the second we have $(n-1)$ objects. At the next to the last step we have only two objects, and the last step there is only one object left. Thus,

$$\begin{aligned} P_n &= n(n-1)\cdots 2\cdot 1 = \\ &= 1\cdot 2\cdots n = n! \end{aligned}$$

Factorials get large very fast: $1! = 1,\ 2! = 2,\ 3! = 6,\ 4! = 24,\ 5! = 120,\ 6! = 720,\ \text{ and } 7! = 5040$.

Example 1: Find the number of ways to arrange 6 cars in the parking lot with 6 places. The answer is $6! = 720$. Example 2: License plates in NC have the form $ABD - 6181$. That is they begin with three letters and end with four digits. Find the number of such license plates with distinct letters. The answer is $26\cdot 25\cdot 24\cdot 10^4$.

2. *Permutations of $r$ objects selected among $n$ distinct objects.*

The definition is the same as earlier, except that we have to arrange the subset of $r$ objects among $n$. The number of such permutations (of the objects among $n$ different objects) is denoted $P_r^n$. Then

$$\begin{aligned} P_r^n &= n(n-1....(n-r+1) = \\ &= \frac{n(n-1)...(n-r+1)(n-r)(n-r-1)...2\cdot 1}{(n-r)(n-r-1)...2\cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

The proof is the same. Let's remark, that

$$P_n = P_n^n = n!$$

This formula means that we have to understand $0! = 1$, i.e.

$$P_n^n = P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Example 3: The number of ways to arrange 6 cars on the parking lot with 15 parking places is equal to

$$P_6^{15} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 = 3,603,600.$$

(15 places for the first car, 14 for the second etc.)

Sometimes the order is not essential. Such unordered arrangements are known as combinations. For 3 distinct objects $A, B, C$ we have $P_2^3 = 3 \cdot 2 = 6$ permutations of the length 2:

$$\begin{matrix} (AB) & (BA) \\ (AC) & (CA) \\ (BC) & (CB) \end{matrix},$$

but only three combinations (the left column in the previous table). The reason is that (A,B) and (B,A) are identical as unordered combinations. The notation for the number of unordered combinations is $\begin{pmatrix} n \\ r \end{pmatrix}$ ("$n$ choose $r$"), other possible notations

$$C_r^n\ ,\ C_n^r\ ,\ C(n,r).$$

Because $r$ distinct objects can be arranged by $r!$ ways, we have

$$\begin{pmatrix} n \\ r \end{pmatrix} \cdot r! = P_r^n\ ,$$

i.e.

$$\begin{aligned} \begin{pmatrix} n \\ r \end{pmatrix} &= \frac{P_r^n}{r!} = \frac{n!}{(n-r)!r!} = \\ &= \frac{n!}{r!(n-r)!} = \begin{pmatrix} n \\ n-r \end{pmatrix}, \\ r &= 0, 1, ..., n\ ;\ \begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ n \end{pmatrix} = 1. \end{aligned}$$

Combinatorial expressions $\begin{pmatrix} n \\ r \end{pmatrix}$ are known also as binomial coefficients, because

$$\begin{aligned} (a+b)^n &= a^n + \begin{pmatrix} n \\ 1 \end{pmatrix} a^{n-1}b + \begin{pmatrix} n \\ 2 \end{pmatrix} a^{n-2}b^2 + ... \\ + \begin{pmatrix} n \\ r \end{pmatrix} a^{n-r}b^r + ... + \begin{pmatrix} n \\ n+1 \end{pmatrix} ab^{n-1} + b^n &= \sum_{K=0}^{n} \begin{pmatrix} n \\ K \end{pmatrix} a^{n-K}b^K. \end{aligned}$$

(this is the famous binomial formula).
Let's remark, that

$$\binom{n}{r} = \binom{n}{n-r}, \binom{n}{n} = \binom{n}{0} = 1,$$
$$\sum_{r=0}^{n} \binom{n}{r} = 2^n \text{ (Put } a = b = 1 \text{ in binomial formula).}$$

In fact

$$\begin{aligned}\binom{n}{r} + \binom{n}{r+1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!} = \\ &= \frac{n!}{r!(n-r-1)!}\left[\frac{1}{n-r} + \frac{1}{r+1}\right] = \frac{n!}{r!(n-r-1)!} \cdot \frac{(n+1)}{(n-r)(r+1)} \\ &= \frac{(n+1)!}{(r+1)!(n-r)!} = \binom{n+1}{r+1}\end{aligned}$$

The equality $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ means that the binomial coefficients behave just like the entries of Pascal's triangle:

$$\begin{array}{ccccccccccccc} &&&&&& 1 &&&&&& \\ &&&&& 1 && 1 &&&&& \\ &&&& 1 && 2 && 1 &&&& \\ &&& 1 && 3 && 3 && 1 &&& \\ && 1 && 4 && 6 && 4 && 1 && \\ & 1 && 5 && 10 && 10 && 5 && 1 & \\ 1 && 6 && 15 && 20 && 15 && 6 && 1 \end{array}$$

In fact they ARE the entries of Pascal's triangle.

These binomial coefficients $\binom{n}{r}$ satisfy the same conditions as the solutions of the following problem: How many ways are there to divide a group of $n$ distinct objects into two subsets, one with $r$ elements and the other with $n - r$ elements. Indeed, to make such a partition we can take a set of $r$ elements from the initial set in exactly $\binom{n}{r}$, ways; the remaining $n - r$ elements are the second subset.

Example 4: Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n} = 2^n.$$

Set $a = b = 1$ in the binomial theorem equation.

Example 5: Prove that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots \binom{n}{2k} \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

Hint: Put $a = 1, b = -1$, and use the binomial formula.

Example 6: Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study at least one of the three subjects?

Solution. Let $M, S$, and $C$ denote the sets of students who study math, science and computing respectively and let $U$ be the entire set of 18 students. Then $|M| = 7, |S| = 10$, and $|C| = 10$. Also, we have $|MS| = 3, |MC| = 4$, and $|SC| = 5$, where, $|x|$ denotes the number of elements of the set $x$ and juxtaposition of sets means intersection. Finally, $|MCS| = 1$. Then

$$|M \cup S \cup C| = (|M| + |S| + |C| - |MS| - |MC| - |SC| + |MSC|)$$

$= 27 - 12 + 1 = 16.$