

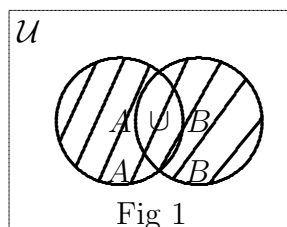
Lecture 6. Operations on Sets

Suppose that \mathcal{U} is a universal set (for given fixed context) and A, B, C, \dots are *subsets* of \mathcal{U} , i.e. $A \subseteq \mathcal{U}$, $B \subseteq \mathcal{U}, \dots$. Using these subsets and operations on sets, one can construct new sets. To understand the nature of operations on sets, it's useful to illustrate definitions with Venn diagrams.

IA. If A, B are two sets (subsets of \mathcal{U}), we *define* their union as a set containing all elements from A or B (or both A and B).

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in \text{both } A \text{ and } B\}.$$

Because of the 'inclusive' meaning of 'or', we often drop the final condition of the definition (since it can be inferred). Fig 1 illustrates the definition.



Union is a natural analogue of summation.

IB. The same definition applies to the union of several sets

$$\begin{aligned} A_1 \cup A_2 \cup \dots \cup A_k &= \cup_{i=1}^k A_k \\ &= \{x \mid x \text{ belongs to at least one of the sets } A_i, i = 1, 2, \dots, k\}. \end{aligned}$$

It is clear that the operation of union does not depend on the order in which the sets appear.

Commutative property:

$$A \cup B = B \cup A.$$

Also, the operation does not depend on the brackets which one can use inside the union of sets.

Associative property:

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

Because there is no way to misunderstand it, we use $A \cup B \cup C$ for either of the two expressions above.

IIA. The next operation is the intersection of two sets. If A and B are given, A intersect B , written $A \cap B$, is defined by

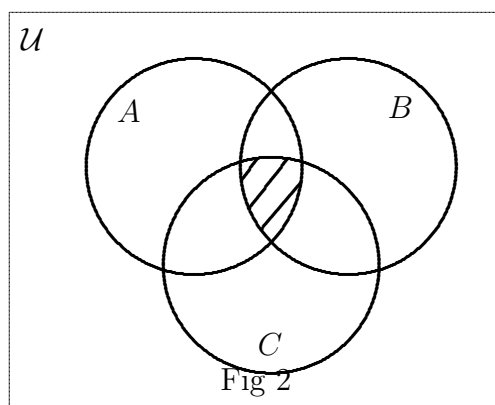
$$A \cap B = \{x \mid x \text{ belongs to both of the sets } A \text{ and } B\}.$$

Of course, we can define the intersection operation for many sets just as we did for the union.

IIB. The operation of intersection for more than two sets. We define the *intersection* of several sets A_1, A_2, \dots, A_k as the set consisting of all elements x , that belong to all the sets A_1, \dots, A_k . In symbols,

$$A_1 \cap A_2 \cap \dots \cap A_k = \bigcap_{i=1}^k A_i = \{x \mid x \in A_1, x \in A_2, \dots, \text{ and } x \in A_k\}.$$

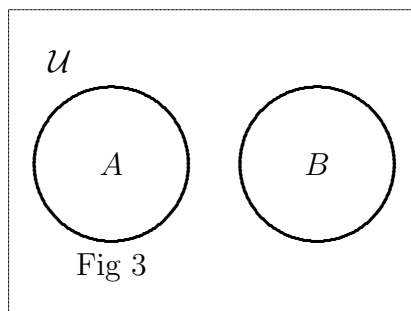
Fig 2 gives an illustration of the definition (for $k = 3$).



The operation of intersection is a natural analogue of multiplication. Instead of the notation \cap , one can use the notation \cdot , i.e. $A_1 \cdot A_2 \cdot \dots \cdot A_k$ means the same as $A_1 \cap A_2 \cap \dots \cap A_k$.

The last two relations are the core elements which connect set theory with the theory of *Boolean algebras* and *logic*, topics we will discuss later in the course.

III. Sets A and B are called disjoint if $A \cap B$ is the empty set ϕ (i.e. $A \cap B$ does not contain any elements). See Fig 3.

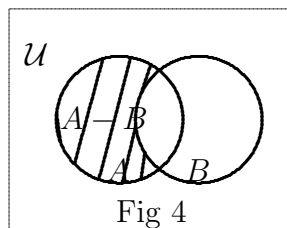


IV. If A and B are two sets, we define the *difference* $A - B$ or the *compliment*

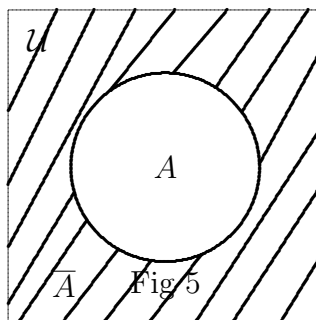
of B with respect to A by the formula

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

See Fig 4.



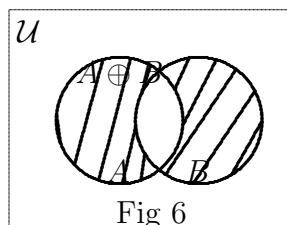
V. If \mathcal{U} is an universal set and $A \subset \mathcal{U}$, then $\mathcal{U} - A$ is called a compliment of A . The notation is $\mathcal{U} - A = \bar{A}$. See Fig 5.



VI. One can also use the special notation $A \oplus B$ for the symmetric difference of the sets A and B . By definition,

$$A \oplus B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}.$$

See Fig 6.



It is clear that

$$A \oplus B = (A \cup B) - (A \cap B)$$

or

$$A \cup B = (A \oplus B) \cup (A \cap B).$$

It is also obvious that

$$(A \oplus B) \cap (A \cap B) = \phi.$$

The properties of set operations

The properties of commutativity and associativity for the operations \cup and \cap are superficial. The *distributive property* combines union and intersection in the same way the distribution property does for multiplication over addition in the system of numbers. Note that there are two distribution properties in set theory but only one in arithmetic.

Intersection distributes over union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Union distributes over intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Properties of the *compliment* are new but simple:

$$\begin{aligned} \overline{(\overline{A})} &= A, & A \cup \overline{A} &= \mathcal{U}, & A \cdot \overline{A} &= \phi, \\ \overline{\phi} &= \mathcal{U}, & \text{and } \overline{\mathcal{U}} &= \phi. \end{aligned}$$

The following properties, known as *De Morgan's laws* (in fact, they are known to Plato and Aristotle in the logical context), are fundamental:

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

It is possible to prove both formulas using Venn diagrams, but one can give a "purely logical" proof. Let us prove property 2.

Step I. Suppose that $x \in \overline{A \cap B}$. Then $x \notin A \cap B$, so $x \notin \{\text{both } A \text{ and } B\}$. This means that either $x \notin A$ or $x \notin B$ or $x \notin A$ and $x \notin B$, i.e. either $x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{A} \cap \overline{B}$. This implies that $x \in \overline{A} \cup \overline{B}$. Conversely, suppose that $x \in \overline{A} \cup \overline{B}$, i.e. either $x \in \overline{A}$ or $x \in \overline{B}$ (or both \overline{A} and \overline{B}). It follows that $x \notin A$, or $x \notin B$, i.e. $x \notin A \cap B \Rightarrow x \in \overline{A \cap B}$. We ask the reader to provide another proof using characteristic functions later in the chapter.

A more general form of the De Morgan's laws is related to a family of sets:

$$\text{A. } \overline{(A_1 \cup A_2 \cup \dots \cup A_k)} = \overline{(\cup_{i=1}^k A_i)} = \overline{A_1} \cdot \overline{A_2} \cdot \dots \cdot \overline{A_k}$$

$$\text{B. } \overline{(A_1 \cap A_2 \cap \dots \cap A_k)} = \overline{(\cap_{i=1}^k A_i)} = \cup_{i=1}^k \overline{A_i}.$$

The proof is the same for these more general properties as it was for properties 1. and 2. above.

There is another operation on sets that is important in the theory of counting and vital in the theory of binary relations. See lecture 9 also. It is called the *cartesian product*. Let A and B be sets. The cartesian product, written $A \times B$ is the set of all ordered pairs whose first element comes from A and whose second comes from B . Symbolically,

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

For example, $\{1, 2\} \times \{a, b, c\} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$. Its not too hard to see that if A and B are finite sets, then $|A \times B| = |A| \cdot |B|$. Also, note that it is generally not true that $A \times B = B \times A$ because, for example the ordered pairs $(1, a)$ and $(a, 1)$ are different. This is why these pairs are called *ordered pairs*.

Indicators or characteristic functions.

If A is a subset of the universal set \mathcal{U} , one can define the function $f_A(x)$ on \mathcal{U} , that is a rule that assigns to each element $x \in \mathcal{U}$ the number 0 or 1 according to formula

$$f_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

This function is called the “*characteristic function*” of the set A or an indicator of A . The notation

$$I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is more popular in mathematical literature than the notation $f_A(x)$ from the text-book. Fig. 8 illustrates the idea of the characteristic function. $\mathcal{U} = [0, 1]$, $A = [1/2, 3/4]$

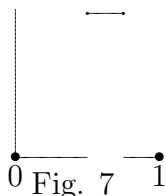


Fig. 7

The following properties of characteristic functions are simple and useful.

1. If $A \subseteq B$, $f_B(x) \geq f_A(x)$.
2. $f_{\overline{A}}(x) = 1 - f_A(x)$, $x \in V$.

Proof:

$$1 - f_A(x) = \begin{cases} 1 - 1, & x \in A \\ 1 - 0, & x \in \overline{A} \end{cases} = \begin{cases} 0, & x \in A \\ 1, & x \in \overline{A} \end{cases} f_{\overline{A}}(x)$$

3. $f_{A_1 \cdot A_2 \cdot \dots \cdot A_k}(x) = f_{A_1}(x) \cdot f_{A_2}(x) \cdot \dots \cdot f_{A_k}(x)$

Proof:

$$f_{A_1}(x) \cdots f_{A_k}(x) = 1 \Leftrightarrow f_{A_1}(x) = 1, \dots, f_{A_k}(x) = 1 \text{ etc.}$$

4. $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x)f_B(x)$

Proof (see 1): Using the De Morgan laws a few times and the relation $f_{\overline{A}}(x) = 1 - f_A(x)$, we have

$$\begin{aligned} f_{A \cup B}(x) &= 1 - f_{\overline{A \cup B}}(x) = 1 - f_{\overline{A} \cap \overline{B}}(x) = 1 - f_{\overline{A}}(x)f_{\overline{B}}(x) \\ &= 1 - (1 - f_A)(1 - f_B) = 1 - (1 - f_A - f_B + f_A \cdot f_B) = f_A + f_B - f_A f_B. \end{aligned}$$

5. $f_{A-B}(x) = f_A - f_A \cdot f_B$.

Proof:

$$\begin{aligned} A - B &= A \cap (\overline{B}) \Rightarrow f_{A-B}(x) = f_{A \cap \overline{B}}(x) \\ &= f_A(x)f_{\overline{B}}(x) = f_A(x)(1 - f_B(x)) = f_A(x) - f_A(x)f_B(x). \end{aligned}$$

6. $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$.

Proof: It follows from the definition that

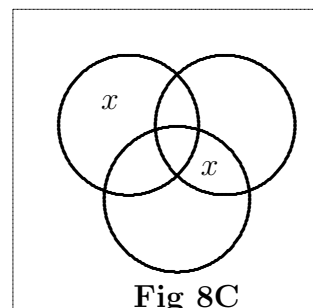
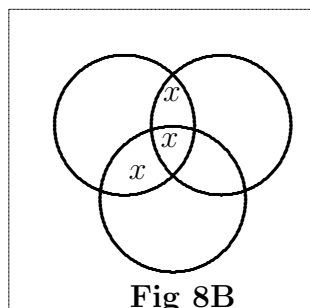
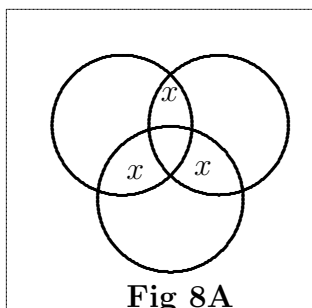
$$A \oplus B = (A \cup B) - AB, \text{ and } AB \subseteq (A \cup B).$$

According to formula 5.

$$\begin{aligned} f_{A \oplus B} &= f_{A \cup B} - f_{AB} = f_A + f_B - f_{AB} = \\ &= f_A + f_B - f_A \cdot f_B = f_A \cdot f_B = f_A + f_B - 2f_{AB}. \end{aligned}$$

Problems

1. Use characteristic functions to prove that \oplus is associative.
2. Using set operations, express the areas with x 's in terms of sets A, B , and C for the following pictures.



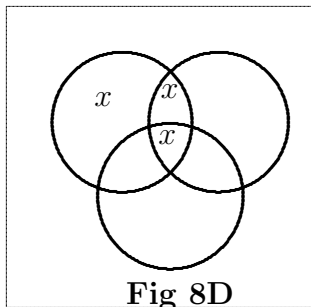


Fig 8D

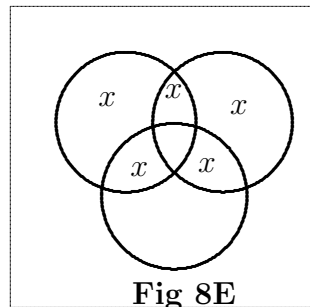


Fig 8E

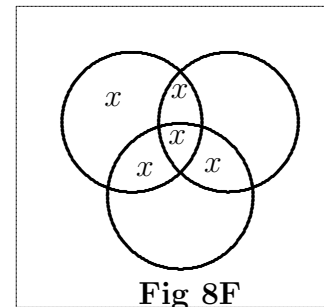


Fig 8F

The answers are not unique.

Set Identities

All sets referred to are subsets of \mathcal{U} .

1. **Commutative Laws:** for all sets A and B ,

- $A \cap B = B \cap A$ and
- $A \cup B = B \cup A$.

2. **Associative Laws:** for all sets A and B ,

- $A \cap (B \cap C) = (A \cap B) \cap C$ and
- $A \cup (B \cup C) = (A \cup B) \cup C$.

Similar formulas are true for a larger number of sets. Writing AB for $A \cap B$, for example, $A(BC)D = (AB)(CD)$ and $A \cup (B \cup C) \cup D = (A \cup B) \cup (C \cup D)$. Thus, the symbolism $A_1 \cup A_2 \cup \dots \cup A_n = \cup_{i=1}^n A_i$ and $A_1 A_2 \dots A_n = \cap_{i=1}^n A_i$ have only one interpretation.

3. **Distributive Laws:**

- $A(B \cup C) = AB \cup AC$ and
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4. **Identities that involve \mathcal{U} and ϕ :**

- $A \cap \mathcal{U} = A$, $A \cup \mathcal{U} = \mathcal{U}$,
- $A \cap \phi = \phi$, and $A \cup \phi = A$.

5. **Double Compliment Law:** $\overline{\overline{A}} = A$

6. **Boolean Laws:**

- $A \cap A = A$ and
- $A \cup A = A$.

7. **Absorption Laws:** If $B \subseteq A$, then $A \cup B = A$ and $A \cap B = B$.

In particular, for all A and B , $A \cup (AB) = A$ and $(A \cup B)A = A$.

8. **DeMorgan's Laws:**

- (a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and
- (b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

9. **Alternative formula for difference:** $A - B = A \cap \overline{B}$.