Lecture 5. Introduction to Set Theory and the Pigeonhole Principle

A set is an arbitrary collection (group) of the objects (usually similar in nature). These objects are called the elements or the members of the set. The set of all real numbers in the interval $[0, 1]$, the set of people in a certain room, the set of the possible remainders modulo 7, the set of prime numbers, and the set of all white elephants, are examples of sets.

For names of sets we use upper case Latin letters $A, B, C, \ldots, Z$ and for the elements we use lower case letters $a, b, c, \ldots x$. The notation

$$x \in A$$

is read "x is an element of A" or simply "x belongs to A".

In this course we deal only with sets that are the parts of some largest (for a given context) set, the so-called *universal set* $\mathcal{U}$.

For instance, in arithmetic, the typical sets are the sets of integers. Therefore, when we work with sets in number theory, we choose $\mathcal{U}$ to be the set of all integers:

$$Z = \{0, \pm 1, \pm 2, \ldots, \pm n \ldots\}.$$

Since we use this set so often, however, it is convenient to give this set its own name, Z. The set of all positive integers is denoted by Z^+:

$$Z^+ = \{1, 2, \ldots n, \ldots\}$$

.

The notation $B \subseteq A$ means, that B is a *subset* of A, i.e. each element b of B also belongs to A. It is convenient to use the notation $B \subset A$ as well. Some authors use the notation $B \subset A$ to mean that B is a *proper* subset of A; that is A has at least one element that B does not have. In Reiter's course, we will not distinguish between $\subseteq$ and $\subset$. They both mean "is a subset of". In Molchanov's course, we make the distinction between the symbols.

Equality of sets. We say two sets A and B are equal, and write $A = B$ if A and B have exactly the same elements. The order in which the elements are listed does not matter. For example $\{1, 2, 3\} = \{3, 1, 2\}$, because they have the same elements.

In calculus, one of the fundamental sets is the set of all real numbers R (also called the real line). It is universal for all possible collections (subsets) of the real numbers. For example, $A = \{x| \quad 0 \leq x \leq 1\} = [0, 1]$ is a subset of R. The symbol "$|$" is read "such that". When the context makes it clear that the numbers in question are real or integers, we often do not explicitly say this. For example, let $A = \{x|\, x \text{ is a real number and } -1 \leq x \leq 1\}$ and let $B = \{x|\, x^2 \leq 1\}$ and $C = \{x|\, x^2 = 1\}$. It is clear that $A = B \supset C$. Of course $B \supset C$ means $C \subset B$.

We will illustrate different definitions and relations in set theory using *venn diagrams.* The universal set here is depicted as a rectangle $\mathcal{U}$, with different subsets depicted as circles inside $\mathcal{U}$:

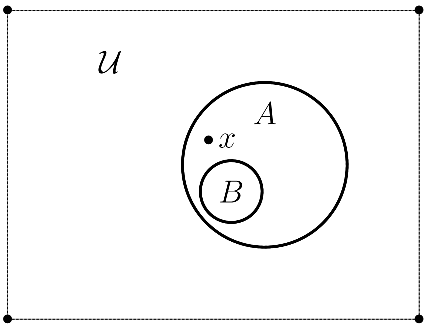


Fig.1 $x \in A$, $B \subset A$, $x \notin B$ (does not belong to B) $A \subset \mathcal{U}$, and $\mathcal{B} \subset \mathcal{U}$

How is it possible to "describe" a set? There are two methods, the listing method and the set description method. We saw both of these at work earlier.

I. Listing method. The set A is given by the "list" of all its elements

$$A = \{x_1, x_2, \ldots x_n\}\,.$$

As you know, the order in which the elements appear is irrelevant. Such a method is important for computers. The list of elements can be "embedded" in the memory of a computer. All constructive sets are *finite*, i.e. contain a finite number of elements. Our convention here is that when listing the elements of a set, we list each element only once. Thus $\{1,2,3,2,3\}$ would be written $\{1,2,3\}$. We'll see later that the two properties we've mentioned, namely that (a), that order is irrelevant, and (b) that multiple membership is not allowed are the properties that distinguish *sets* from *sequences.*

II. Description method. Set A can be defined as a group of objects x for which some proposition $P(x)$ (sentence, statement) is true. The notation is

$$A = \{x|\ P(x) \text{ is true}\},$$

is used. As noted earlier, the vertical bar "|" is read "such that". *Example.* Let $A_7 = \{x|\, x \text{ is a possible remainder upon division by } 7\}$. This set is given descriptively. The set could be listed as well:

$$A_7 = \{0,1,2,3,4,5,6\}\,.$$

Sometimes the transition from the descriptive representation to the listing is not trivial.

Example. $A = \{x|\, x \text{ is a real root of the cubic equation } x^3 - 2x^2 - 7x + 14 = 0\}$. If you solve this equation (using, for instance, factorization) you'll get the listed form of the same set

$$A = \left\{2,\ \sqrt{7},\ -\sqrt{7}\right\}.$$

Example. The set $P = \{x|\, x \text{ is a prime number less than } 10^{20}\}$ is an example of a set that is much more easily described than listed. For sets such as this one, there is an algorithm which leads to a listing (sieve of Erathosthenes, see lecture 1), but its realization is beyond the capabilities of modern computers. Thus the set P today has only the descriptive definition.

Cardinality and The Set Equivalence Theory

Functions Let A and B be sets. A function *From A to B* is a subset f of $A \times B$ satisfying the two conditions

1. each element of A is the first coordinate of some ordered pair of f, and
2. No two ordered pairs of f have the same first coordinate.

In this case the set A is called the *domain* of f. If $(a, b) \in f$ we simply write $f(a) = b$. With this notation, condition 1. says that for each $a \in A$, there is a $b \in B$ such that $f(a) = b$, and condition 2. says that for each $a \in A$, there is just one $b \in B$ such that $f(a) = b$. Taken together, the two conditions say that for each $a \in A$ there is a unique $b \in B$ such that $f(a) = b$.

Example 1. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Let

1. Let $f = \{(1, a), (2, a), (3, b)\}$. Note that each element of A is the first coordinate of some ordered pair and that no two ordered pairs have the same first coordinate, so f is a function from A to B.
2. Let $g = \{(1, a), (2, b)\}$. This subset of $A \times B$ is a function, but it is not a function from $\{1, 2, 3\}$ to B.
3. Let $h = \{(1, a), (1, b), (2, c), (3, c)\}$. This subset of $A \times B$ is not a function because it has two ordered pairs $(1, a)$ and $(1, b)$ with the same first coordinate.

We need a few more definitions before we can discuss equivalence of sets. A function f from A to B is called

1. *One-to-one* (injective) if no two members of f have the same second coordinate. Another way to say this is to say that if a_1 and a_2 are different members of A, then $f(a_1) \neq f(a_2)$. Otherwise there would be two ordered pairs with the same second coordinate. Such functions are also called injections.

2. *Onto* (surjective) if every element of B is the second coordinate of some member of f. Such functions are also called surjections.

3. A *set equivalence* (or a bijection) if it is both one-to-one and onto. Such functions are also called one-to-one correspondences.

   Example 2. Again let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Let $k = \{1, a), (2, b), (3, c)\}$. The function k is one-to-one because no element of B appears more than once and onto because each element of B appears once. Thus, k is a set equivalence between A and B. We use the notation $A \approx B$ when A and B are equivalent; ie, when there is a set equivalence from A to B. We also use the notation $[n] = \{1, 2, 3, 4, \ldots, n\}$, where n is a positive integer. Thus $[n]$ is the set of the first n positive integers.

   The following proposition is the basis for the idea of cardinality. You'll see later that $\approx$ is what is called and equivalence relation, and the partition it determines on a collection of sets is what we mean by cardinality of sets.

   Proposition. Let A, B, and C be sets. Then

   (a) $A \approx A$,

   (b) If $A \approx B$, then $B \approx A$, and

   (c) If $A \approx B$, and $B \approx C$ then $A \approx C$.

   Proof. The first part follows because the identity function from and set A to itself is a set equivalence. To prove the second, suppose $f : A \to B$ is an equivalence. Then the inverse relation $f^{-1} = \{(b, a) : (a, b) \in f\}$ is a function from B to A because each element $b \in B$ appears exactly once as a first coordinate of f^{-1}. It also is true that f^{-1} is one-to-one and onto as well, so f^{-1} is a bijection. Thus $B \approx A$. To prove the third part, suppose $f : A \to B$ and $g : B \to C$ are bijections. Then the composition relation $g \circ f$ is a function. We leave it as an exercise to show that $g \circ f$ is a bijection. This $A \approx C$.

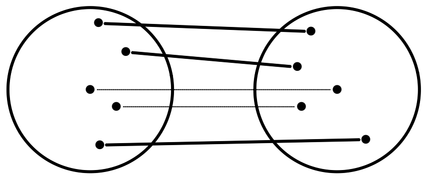
   A set A is called *finite* if either A is the empty set or $A \approx [n]$ for some positive integer n. In this case we write $|A| = 0$ or $|A| = n$. If $A = \{x_1, \ldots, x_n\}$, containing n distinct elements, the number n is called the cardinality of A. Notation. We use $|\ |$ on finite sets to mean the number of members.

   $$n = \text{Card}(A) = |A|.$$

   Let us repeat that there are several ways to say that two sets A and B are equivalent. We could say they have the same cardinality, that there is a one-to-one correspondence between them, that the are equinumerous, or that $|A| = |B|$.

Thus, if A and B are finite sets and $B \subseteq A$, then $|B| \leq |A|$. Furthermore, if B is a proper subset of A, then $|B| < |A|$.

The typical picture has the form:



There is a one-to-one correspondence between the set of students in my class and the corresponding names (or ID numbers) on the roster.

It is obvious that the existence of a one-to-one correspondence between elements of two finite sets A and B implies that $|A| = |B|$. The converse statement is also correct: if $|A| = |B|$, we can enumerate their elements:

$$\begin{aligned} A &= \{x_1, x_2, \ldots, x_n\} \ , \ n = |A| \\ B &= \{y_1, y_2, \ldots, y_n\} \ , \ n = |B| = |A| \end{aligned}$$

and the desired correspondence has the form: $x_1 \leftrightarrow y_1,\ x_2 \leftrightarrow y_2, \ldots, x_n \leftrightarrow y_n$.

The set is *infinite* if it contains an infinite number of elements. The simplest examples of infinite sets are the *countable sets*, which are those sets for which there is one-to-one correspondence with the set $Z^+ = \{1, 2, 3, \ldots\}$ of all positive integers.

For instance, the set $O = \{1, 3, 5, \ldots, 2k-1, \ldots\}$ of positive odd numbers and the set $E = \{2, 4, 6, \ldots, 2k, \ldots\}$ of positive even numbers are countable. One-to-one correspondences are straightforward in both cases:

$$\begin{aligned} k &\leftrightarrow 2k-1,\ k = 1, 2, \ldots \\ k &\leftrightarrow 2k,\ k = 1, 2, \ldots \end{aligned}$$

Let's stress that $E \subset Z^+, O \subset Z^+$ are proper subsets of Z^+. Yet each of these sets can be viewed as equal in size with Z^+. This is the common feature of all

infinite sets. Note that for finite sets, proper subsets must have fewer elements. Here are two more examples. Recall the N denotes the set of natural numbers, $N = \{0, 1, 2, \ldots\}$. It is perhaps the most commonly encountered infinite set.
Proposition 1. $N \approx Z^+$. **Proof.** To prove that $N \approx Z^+$, we must produce a one-to-one function from N onto Z^+. Let $f : N \to Z^+$ be defined by $f(n) = n + 1$. Now clearly f is a function with domain N. To see that f is one-to-one, suppose $f(n) = f(m)$. Them $n + 1 = m + 1$ and $n = m$. To see that f is onto, let $m \in Z^+$. Then $f(m - 1) = m - 1 + 1 = m$.

Proposition 2. The sets Z^+ and Z are equivalent. First we need to define the function from Z^+ to Z. Let

$$f(x) = \begin{cases} -x/2 & \text{if } x \text{ is even} \\ (x-1)/2 & \text{if } x \text{ is odd} \end{cases}$$

To prove that f is a bijection, we must show that it is one-to-one and onto. To see that it is one-to-one, take two different positive integers, x and y. If x and y are both even, then $f(x) = -x/2 \neq -y/2 = f(y)$ because if they were equal, then x would be equal to y. Similarly if x and y are both odd. Now if one is even and the other odd, then their functional values have different signs, and so cannot be equal. To see that f is onto, let b be a non-negative integer. Then $2b + 1$ satisfies $f(2b + 1) = (2b + 1) - 1 \div 2 = 2b \div 2 = b$. On the other hand, suppose b is a negative integer. Then $-2b$ is the number we need: $f(-2b) = -(-2b) \div 2 = b$. Thus f is a one-to-one correspondence (ie, a bijection).

More challenging problems. We now show how to make a point disappear.
Proposition 3. The closed unit interval $[0, 1]$ is equinumerous with the half closed unit interval $[0, 1)$.
Proof. Define the function $f : [0, 1] \to [0, 1)$ as follows:

$$f(x) = \begin{cases} 2^{-n-1} & \text{if } x = 2^{-n} \text{ for some positive integer } n \\ x & \text{otherwise} \end{cases}$$

Clearly, f is well-defined on $[0, 1]$. For convenience, let $R = \{1, 1/2, 1/4, 1/8, \ldots\}$. So f can be described as the function that takes half of each element of R and leaves all the other elements of its domain fixed. To see that f is one-to-one, let u and v be different numbers in $[0, 1]$. We consider three cases. Case 1, $u, v \in R$. In this case the smaller of u and v has the smaller functional value. Case 2, one of the two, say u, is in R and the other is not. In this case $f(u) \in R$ while $f(v)$ is not in R. Case 3, neither is in R. In this case $f(u) = u \neq v = f(v)$. To see that f is onto, suppose $y \in [0, 1)$. If $y \in R$ then, $2y \in R$ and $f(2y) = y$. If y is not in R, the $f(y) = y$.

Note that every infinite sequence of distinct integers is countable. That is the set of all the elements of an infinite sequence is countable. For Example. the set S of perfect squares $\{1, 4, 9, \ldots, k^2, \ldots\}$ and the set F of factorials $\{1, 2, 6, 24, \ldots k!, \ldots\}$ can be put into one-to-one correspondence with the set Z^+.

If we use the notation $A \Leftrightarrow B$ to mean that there is a one-to-one correspondence between A and B, then the following properties of $\Leftrightarrow$ can be proven.

(a) Reflexive property: for all sets A, $\quad A \Leftrightarrow A$.

(b) Symmetric property: for all sets A and B, if $A \Leftrightarrow B$, then $B \Leftrightarrow A$.

(c) Transitive property: for all sets A, B, and C, if $A \Leftrightarrow B$ and $B \Leftrightarrow C$, then $A \Leftrightarrow C$.

The theory of infinite sets with applications to calculus and logic was developed by the German mathematician *George Cantor*. *Theorem* (Cantor). Let $Q = \left\{ v \in R : v = \frac{m}{n}, m,\ n \text{ are integers} \right\}$ be the set of all fractions (positive or negative). *The set* Q is *countable.*

It looks very strange, because the set Q is "dense" in the real line R : between two arbitrary real numbers $x, y :\ x < y$ one can find a rational number.

Proof of the theorem. We can suppose that m and n have no non-trivial common divisor $(GCD(m, n) = 1), n \geq 1,\ m \in Z$ and use the notation $\frac{m}{1}$ for integers (which are also rational numbers).

Let $h(v) = h(\frac{m}{n}) = |m| + |n|$ be the "height" of the fraction v. The crucial observation is that there exists only finite number of fractions with the given fixed

	$n = 1$	$0/1$								
	$n = 2$	$1/1$	$-1/1$							
height n :	$n = 3$	$2/1$	$1/2$	$-1/2$	$-2/1$					etc.
	$n = 4$	$3/1$	$1/3$	$-1/3$	$-3/1$					
	$n = 5$	$4/1$	$3/2$	$2/3$	$1/4$	$-1/4$	$-2/3$	$-3/2$	$-4/1$	

Now we can enumerate all rational numbers moving along rows with increasing heights from left to right within each row. So 0 is first, 1 is second, -1 is third, 2 is fourth, etc.

Definition. A number α is *algebraic* if it satisfies an algebraic equation of some degree $m \geq 1$ with integer coefficients:

$$\begin{aligned} a_m\alpha^m + a_{m-1}\alpha^{m-1} + \ldots + a_1\alpha + a_0 &= 0 \\ a_m, a_{m-1}, \ldots, a_0 &\in Z,\ m \geq 1. \end{aligned}$$

All rational numbers are algebraic: if $r = \frac{m}{n}$, then r is the root of the linear ($m = 1$) equation $n \cdot r - m = 0$. There are many irrational algebraic numbers:

$$\begin{aligned} \alpha &= \sqrt{2}, \text{ root of } \ x^2 - 2 = 0 \\ \alpha &= \sqrt[5]{1 + \sqrt[3]{7}}, \text{ root of } (x^5 - 1)^3 - 7 = 0 \text{ etc.} \end{aligned}$$

Theorem (Cantor). The set of all real algebraic numbers is countable.

The sketch of the proof. If $P_m(x) = a_m x^m + \ldots + a_1 x + a_0$ is a polynomial of the degree m with integer coefficients, the "height" of $P_m(x)$ is given by formula

$$h(P_m) = m + |a_m| + \ldots + |a_1| + |a_0|.$$

For each $h = 2, 3, \ldots$ there exists only a finite number of such polynomials:

$$\begin{aligned} h &= 2\ , \ \ P_1(x) = +1 \cdot x\ , \ \ P_1(x) = -1 \cdot x\ , m = 1 \\ h &= 3\ , \ \ P_2(x) = \pm 1 \cdot x^2\ , \ \ P_1(x) = \pm x \pm 1\ , \ P_1(x) = \pm 2 \cdot x \\ & \quad\ \text{(8 different polynomials)} \end{aligned}$$

etc.

Equation $P_m(x) = 0$ has at most m real roots. One can enumerate the algebraic numbers in the following way: for each $h = 2, 3, \ldots$ write out all polynomials of the height h, to find corresponding real (algebraic) root and step by step ($h = 2,\ h = 3, \ldots$) enumerate these roots removing some of them in the case of repetitions.

$$\alpha_0 = 0,\ \alpha_1 = 1,\ a_2 = -1, \ldots$$

After all the results above we might think that all infinite sets are countable! It is not true. There are larger infinities than the infinity of the natural numbers. The procedure below is called the Cantor Diagonalization Procedure. Before we discuss this procedure, let us play a game called Dodgeball. There are two players, A, the aggressor, and B, the dodger. Player A fills out the rows of a 6×6 grid with X's and O's, one at a time while player B fills out one array of squares, one square at a time, also with X's and O's. Player B wins if he can avoid constructing any of player A's rows. The sample game below shows A's first move and B's response to it.

X	X	O	O	O	X

O					

Play this game a few times with a friend. Can you devise a winning strategy for either player? Once you see a winning strategy, you're ready to understand Cantor's procedure. The authors are grateful to Michael Starbird for suggesting this beautiful game.

Theorem (Cantor) The set of real numbers in the open interval (0,1) is not countable (has "continuum" cardinality).

We can understand real numbers $0 < x < 1$ as an infinite series:

$$x = .e_1 e_2 \dots e_n \dots_2 = \frac{e_1}{2} + \frac{e_2}{2^2} + \dots + \frac{e_n}{2^n} + \dots$$

where $e_i = 0$ or $1,\ i = 1, 2 \dots$ (binary representation, one can use decimals). Such representation is not quite unique because every number that repeats zeros from some point has a representation that repeats all 1's from some point as well. For example, $0.11\overline{0}_2 = 0.10\overline{1}_2 = 3/4$. However, this ambiguity is nothing more than a minor annoyance.

Proof by contradiction. Suppose that enumeration of all real numbers in (0,1) is possible. If such a list exists, there must also be a list of those real numbers that have unambiguous binary representations since we could create such a list by first listing all the numbers and then removing the unwanted ones (these ambiguous numbers are called diatic rationals, by the way). Such a "list" must have the form

$$x_1 \quad = \quad 0.a_{11}\, a_{12} \dots a_{1n} \dots$$

$$
\begin{aligned}
x_2 &= 0.a_{21}\, a_{22} \ldots a_{2n} \ldots \\
&\ldots \\
x_n &= 0.a_{n1}\, a_{n2} \ldots a_{nn} \ldots \\
&\ldots
\end{aligned}
$$

Let's consider the entries along the diagonal, i.e. the sequence of the digits

$$a_{11},\ a_{22}, \ldots, a_{n,n}, \ldots$$

. We construct a real number b by specifying its binary digits:

$$b = .b_1 b_2 \ldots b_n \ldots$$

where $b_1 \neq a_{11},\ b_2 \neq a_{22}, \ldots b_n \neq a_{n,n}, \ldots$. That is, $b_k = 0$ if $a_{kk} = 1$ and $b_k = 1$ if $a_{kk} = 0$.

The number b does not appear in the list. First note that it is possible that b is a diatic rational number. (Try to arrange a sequence x_n so that the resulting b has value 1/8). In this case b does not appear in the list. On the other hand if b is not a diatic rational, the following shows that b is not in the list. If it did, it would be identical to one of the numbers $x_1, x_2, x_3, \ldots$. Suppose $b = a_k$. Look at the kth digit in b and x_k. Since $b_k \neq a_{kk}$, it follows that $x_k \neq b$.

$$
\begin{aligned}
b &= x_k = .a_{k1} \ldots a_{kk} \ldots \\
&= .b_1 \ldots b_k \ldots
\end{aligned}
$$

This contradiction proves the theorem. The brilliant idea of using the diagonal of the table is known as Cantor's diagonal method.

The Pigeonhole Principle The Pigeonhole Principle (PHP) is our model of how to prove that certain 'overlap' between two sets exists. Here's an example. Suppose there are 13 students in a class. Must there be two with the same birth month. Of course, we say. But how do we prove such an assertion? The answer is a formal method provided by PHP. Suppose each member of an $n + 1$ element set is assigned to one of n pigeon holes. Then one of the holes must have at least two members of the set assigned to it. Thus the 13 class members are the 'pigeons' and the 12 months are the pigeon holes. The PHP then asserts that there must be at least two students with the same birth month.

Another formulation of the principle says that if A and B are finite sets with $|A| > |B|$, then there is not one-to-one function from A into B. Next we look at a sequence of examples each of which makes use of PHP.

(a) Let A be a set of five lattice points in the plane, that is, points both of whose coordinates are integers. Must the midpoint of some pair of them be a lattice point as well? Some experimentation leads to a 'yes' conclusion. But how can we prove it? Solution. What does it take to guarantee that two lattice points have a midpoint that is also one. In order that $((x_1 + x_2)/2, (y_1 + y_2)/2)$ be a lattice point, both $x_1 + x_2$ and y_1+y_2 must be even numbers. This means that x_1 and x_2 must either both be even or both be odd, and the same is true for y_1 and y_2. Therefore, lets use as pigeon holes the categories (odd, odd), $(odd, even)$, $(even, odd)$, and $(even, even)$. For example, $(2, 3)$ would be put in category(hole) $(even, odd)$. Now, one of the four pigeon holes must have at least two points(pigeons), and those two have a lattice point midpoint.

(b) Prove that every set S of 51 numbers in the set $\{1, 2, 3, \ldots, 100\}$ must contain a pair of consecutive integers. Solution. Again the members of S are the pigeons. Let $\{1, 2\}; \{3, 4\}; \ldots \{99, 100\}$ be the pigeon holes. When the pigeons are distributed, some hole gets two pigeons, and those two pigeons (numbers) are consecutive.

(c) Color all the points of the plane red or blue. Then there must be a pair of monochromatic(one color) points that are exactly distance 1 apart. Solution. Let T be an equilateral triangle with vertices A, B, and C that are one unit apart. Then A, B and C are the pigeons and *red* and *blue* are the pigeon holes.

(d) Let S be a subset of $\{1, 2, 3, \ldots, 99\}$ such that $|S| > 50$. Must there be two members of S whose sum is 100? Solution. The pigeons are the members of S and the holes are $\{1, 99\}; \{2, 98\}; \{3, 97\}, \ldots, \{49, 51\}$; and $\{50\}$. There are 50 holes and more than 50 pigeons, so PHP applies to say that some hole contains two or more numbers. These two must have a sum of 100.

(e) Prove that for any five points in the square with vertices $(-1, -1)$, $(-1, 1), (1, -1), (1, 1)$, there must be two which are no farther apart than $\sqrt{2}$. Solution. Let the four unit squares be the pigeons. Then two of the five points must belong to some unit square, so they cannot be farther apart than $\sqrt{2}$.

(f) Show that any 51 element subset of $\{1, 2, 3, \ldots, 100\}$ must have a pair of numbers a, b such that $a|b$. Solution. Write each number a in S as a product $a = 2^i \cdot u$ where u is an odd number (called the *odd part* of a). How many odd parts are there? The possible odd parts are $1, 3, 5, 7, \ldots, 99$, of which there are 50. Hence, two members of S have the same odd part. Let $a = 2^i \cdot u$ and $b = 2^j \cdot u$, where $i < j$. Then $a|b$ because $b/a = 2^j \cdot u/2^i \cdot u = 2^{j-i}$.

Homework.

(a) (Molchanov's class only) Using Cantor's idea of the "height," prove that the set Q_2 on the plane R_2 containing all points (r_1, r_2) with both rational coordinates is countable.

(b) Construct the one-to-one correspondence between (0,1) and an arbitrary open interval (a,b). Construct similar correspondence between (0,1) and $R_1 = (-\infty, +\infty)$

(c) Construct one-to-one correspondence between $[0, 1]$ and $(0, 1)$.

(d) (Reiter's class only) Construct one-to-one correspondence between $[0, 1]$ and $[0, 1] \times [0, 1]$.