Lecture 11: Optimization on Graphs

In this lecture we'll be discussing two of the most important optimization problems on the graphs. We'll illustrate basic ideas and algorithms on the small graphs (as we did already in the case of Warshall algorithm). The real-life problems may involve many thousands of the edges and vertices (telephone and computer networks, air traffic etc.). It is necessary in such situations to use the computers and algorithms with minimal number of operations. The difference between algorithms requiring $O(n^2)$ and $O(n^3)$ operations for the graph with n vertices can be crucial if n has order $10^4 - 10^6$.

Shortest path problem (simplest case). Suppose that a directed graph $\lceil$ with matrix $M_\lceil$ has all edges of the same length, say $l = 1$. Suppose also that the graph $\lceil$ is connected, i.e. all elements of the matrix $M_{\lceil\infty}$ are equal to 1. In different terms, for arbitrary pair of the vertices $a_i, a_j \in A$ (set of the graph) one can find path $a_1, x_1, x_2, ..., x_{K-1}, a_j$ such that the pairs $(a_i, x_1), (x_1, x_2), ...(x_{K-1}, a_j)$ are connected by $\lceil-$ edges. Then K (number of edges) is the length of the specific path. The problem is among all paths between a_i and a_j to find the shortest one (maybe, the shortest path is not unique).

The simplest and widely used algorithm, so-called Moor algorithm is based on the old physical idea: Huygens's principle. According to this principle, the propagation of the light has the following structure: if we know at some moment t the wave front, then each point on this front can be considered as a new source of radiation (with the same optical properties: speed, frequency, etc.). Wave front at the moment $(t + s)$ will be an envelope of the wave fronts of these sources (fig.1)

Fig 1.

Let's return to the Moor algorithm and fix the initial point a_{i_0}. Label a_{i_0} with 0. Find all unlabeled vertices connected to the vertex a_{i_0} and label these vertices with 1. For each vertex a_i labeled with 1 find all neighboring unlabeled vertices (i.e. vertices at the distance $l = 1$) and label them with 2 and so forth.

After finite number of the steps the process of labeling will reach the second fixed point a_{j_0} and it will get some label $K \geq 1$. It is clear, that the shortest path between a_{i_0} and a_{j_0} has the length k and to find this path, one can take an arbitrary sequence of the points with labels $0, 1, 2, ..., k$.

Example. Let's consider the symmetrical graphs, i.e. graphs with indirect edges:

We can also remove all edges from a_i to a_i, which action cannot change the shortest paths between vertices. Suppose that specific graph is presented by Fig. 2.

The picture shows the labels and the shortest path. To find this path, we have to go backward from j_0 to i_0 along decreasing sequence of the labels: $4 \to 3 \to 2 \to 1 \to 0$.

For the point labeled 5 on the fig. 2 there are two shortest paths from i_0. You have to find these paths.

If graph $\lceil$ has N edges, the total number of operations to construct labeling for a fixed initial point a_{i_0} has order $N(\leq N)$.

The Moor algorithm is natural for the cases when the "physical" length of the edges is irrelevant (electrical circuits, telephone communications, etc.). In most applications the edges (i, j) between a_i and a_j (from a_i to a_j) have any length l_{ij}, generally speaking $l_{ij} \neq l_{ji}$.

If $\lceil$ is a connected graph, i.e. $M_{\lceil \infty} = E = \{e_{ij} \equiv 1\}$ and a_{i_0}, a_{j_0} are two fixed edges, we have a problem of finding the shortest path from a_{i_0} to a_{j_0}, i.e. the distance.

$$L_{i0,j0} \equiv \underbrace{K, x_1, ...x_{K-1}} \min \left(l_{i_0 i_1} + l_{i1i2} + ... + l_{i_{K-1}iK} \right)$$

where $i_0, i_1, ...i_K$ are the numbers of the vertices $a_{i_0}, x_1, ...x_{K-1}, x_K = a_{j0}$ along the path

$$a_{i_0} \longrightarrow x_1 \longrightarrow x_2 \longrightarrow ...x_{K-1} \longrightarrow x_K = a_{j_0}.$$

A corresponding algorithm is based on the Bellman's principle (dynamic programming principle:

If $\pi = (i_0, i_1, ..., i_{K-1}, i_K)$ is a shortest path from i_0 to j_0 on $\lceil$ and (i_{K-1}, i_K) is the last edge of this path, then $\widetilde{\pi} = (i_0, i_1, ...i_{K-1})$ is the shortest path from a_{i0} to x_{K-1}.

This proof is simple. Suppose that the conclusion is false. Then one can find the path $\overline{\pi}$ from i_0 to x_{K-1} that is shorter than $\widetilde{\pi}$. Hence, if we add edge (x_{K-1}, a_{j0}) to the $\overline{\pi}$ we'll obtain a path from a_{i0} to a_{j0} that is shorter than π. Contradiction.

The Bellman's principle gives for the distances $L_{i0j} = L_j$ (for fixed $i0$ and different j) the following non-linear equation:

$$\begin{aligned} L_{i0} &= L_{i_0 i_0} = 0 \\ L_j &= \underbrace{z : j \to z}_{\min}(L_z + l_{zj}) \end{aligned}$$

There are different numerical methods to solve Bellman equation (linear programming etc.).

I'll give a description of the Dijkstra's algorithm, popular in applications. At each stage of computations, each vertex x receives one label from the choice of two:

(PL) a permanent label = length L_x of the shortest path $\pi_{i_0} x$ from a_{i_0} to x.

or

(TL) a temporary label = upper bound $\widetilde{L}_x$ for the L_x.

At the initial step, all vertices x connected by edges with a_{i_0} get the (TL) $L_{i_0} = L_x$ and i_0 gets the (PL) $L_{i_0} = 0$. All other edges y get (TL) $Ly = +\infty$.

At the second step find all x_0 for which L_x is minimum and transfer x_0 from the set with (TL) to the set (PL). Put $L_{x_0} = l_{i_0} x_0$, $x_0 \in$(PL) and for $y \in$(TL) put

$$\widetilde{Ly} = \underbrace{K} \min \left\{ \widetilde{Ly}, L_{x_0} + l_{x0j} \right\}$$

Continue the process.

<u>Example</u>. Consider the following symmetrical graph (without edges $i \to i$).

Initial point: (1).

Step 1. $L_1 = 0,\ \widetilde{L}_2 = 2,\ \widetilde{L}_3 = 8, \widetilde{L}_4 = 5$

$$PL = \{1\}\,,\ TL = \{2,3,4\}$$

$$\widetilde{L}_2 = \min(\widetilde{L}_2, \widetilde{L}_3, \widetilde{L}_4) = 2, x_0 = \{2\}$$

Step 2.

$$PL = \{1,2\},\ TL = \{3,4\}$$

$$L_1 = 0,\ L_2 = 2, \widetilde{L}_3 = \min\{8, 2+3\} = 5$$

$$\widetilde{L}_4 = \min\{5, 2+4\} = 5$$

Step 3. $\min \widetilde{L}_3, \widetilde{L}_4 = 5$

Shortest paths:

1) $L_1 = 0,\ \widetilde{L}_2 = 8,\ \widetilde{L}_3 = 4,\ \widetilde{L}_4 = 2,\ \widetilde{L}_5 = \infty$

$$PL = \{1\}\ TL = \{2,3,4,5\}$$

2) $\min(\widetilde{L}_2, ...\widetilde{L}_4\} = 2 = L_4$

$$PL = \{1,4\}\ TL = \{2,3,5\}$$

$$L_1 = 0,\ L_4 = 2,$$

$$\widetilde{L}_2 = \min(8, 2+2\} = 4$$

$$\widetilde{L}_3 = \min(4, 2+1) = 3$$

$$\widetilde{L}_5 = \min(\infty, 2+7) = 9.$$

3) $\min \widetilde{L}_2, \widetilde{L}_3, \widetilde{L}_5 = 3 = L_3$

$$PL = (1,4,3)\ TL = (2,5)$$

$$\widetilde{L}_2 = (\min(4,\ 2+1+3,\ 2+2) = 4$$

$\widetilde{L}_{15} = \min(3+5,\ 2+7) = 8$

4) $\min \widetilde{L}_2, \widetilde{L}_5 = 4 = L_2 = 4$

$$\begin{aligned} PL &= (1,4,3,2) \\ L_1 &= 0, L_4 = 2, L_3 = 3, L_2 = 4, TL = 15 \\ L_5 &= \widetilde{L}_5 = \min(3+5,\ 2+7) = 8 \end{aligned}$$