Lecture 10. Composition of Relations and Directed Graphs.

The definition of the composition of two (or more) relations or graphs is similar to the definition of composite functions. If $y = f(x),\ x = y(z)$ are two functions and $Rang(g) \subset Dom(f)$ we introduce the (composite) function

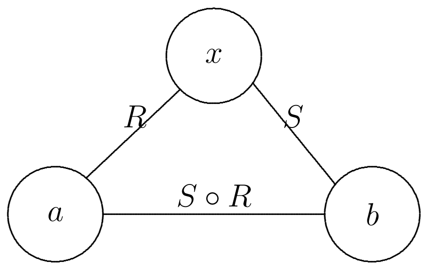
$$y = f(g(z))$$

which acts from $Dom(g) \to Rang(f)$.

Suppose that $A = (a_1, ..., a_n)$ is a set of elements and R, S are two relations on A. The relation $S \circ R$, the *composition* of R and S, has the following definition: $a\ (S \circ R)\ b \Leftrightarrow$ one can find intermediate element x in A such that

$$a\ Rx\ ,\ x\ S\ b$$

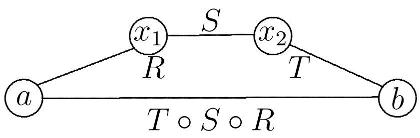
In the language of the graphs Γ_R, Γ_S composition means that we can reach b from a in two steps: an R-step from a to some element x and then an S-step from x to b.



Operation $S \circ R$ is similar to multiplication. One can check that this operation is *associative*, i.e.

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

The explanation is simple: both parts reflect the relation between $a,\ b \in A$ with two intermediate elements $(x_1, x_2 \in A)$:



Generally speaking the operation of composition is not *commutative*, i.e. in general $R \circ S \neq S \circ R$.

Example. $A = (a_1, a_2)$, graphs Γ_R and Γ_S corresponding to relations R and S have a form

R, Γ_R S, Γ_S

Then

$\Gamma_{S \circ R}$ $\Gamma_{R \circ S}$

and

$$\Gamma_{S \circ R} \subset \Gamma_{R \circ S}$$

Especially important case is the composition of the relation R with itself (this operation may be repeated several times):

$$\begin{aligned} R^2 &= R \circ R,\ R^3 = R \circ R \circ R = R^2 \circ R = R \circ R^2, \\ R^k &= \underbrace{R \circ R \circ \ldots \circ R}_{\kappa \text{ times}} = R^{n-1} \circ R = R \circ R^{n-1}. \end{aligned}$$

In the language of graph theory, the definition of the "*degree*" for a relation R is very simple. Given directed graph Γ, i.e. a system of vertices $A = \{a_i, i = 1, \ldots k\}$

and directed edges (links) between some of the pairs (a_i, a_j), *path of length* k in Γ from a to b is a finite sequence

$$a\ ,\ x_1,\ x_2, \ldots,\ x_{k-1},\ b$$

such that

$$a\,R\,x_1,\ x_1\,R\,x_2, \ldots\ x_{k-1}\,R\,b$$

Length k of the path is simply the number of sequential edges between a and b:

We can demonstrate it by the following relation R between U.S. cities: the existence of a direct flight from a to b. In this case, the relation R^k between a and b means that there exists a sequence of flights from a to b with $(k-1)$ stops.

A path that begins and ends at the same vertex $a \in A$ is called *cycle.* Cycles play a fundamental role in the theory of algorithms. Many practical calculations are based on the cyclical (iterative) algorithms.

Together with relations $R,\ R^2, ...,\ R^k, ...$ one can introduce a relation R^∞, the so-called *connecting relation (or connectivity relation) on* A. By the definition $a\ R^\infty b \Leftrightarrow$ there is some path from a to b (no restrictions on the length of the path).

Theorem. Let A be a finite set with $|A| = n$ and R is a relation on A. Then

$$R^\infty = R \cup R^2 \cup ... \cup R^n.$$

Proof. It is clear from the definition, that $R^\infty = R \cup R^2 \cup ... \cup R^n \cup R^{n+1} \cup ... = \bigcup_{k=1}^\infty R^k$. We can prove that we can find R^∞ by taking the union of only the first n powers of R. That is, $R^{n+l} \subseteq R \cup R^2 \cup ... \cup R^n$ for $l \geq 1$. This means that if there exist a path of the length $(n+l) > n$ between two points $a_1 b \in A$ we can construct a shorter path between the same points (with length $\leq n$).

A path between a and b can contain cycles: . We can make such a path shorter by simply eliminating the cycles from it. (see fig 1) and We simply connect the paths from a to x and from x to b together. This procedure shows that the existence of the path from a to b implies the existence of the path with distinct elements (i.e. path without cycles). But such a path cannot be longer than n. Furthermore, if $a \neq b$, the shortest path between a and b has the length $l \leq n-1$. Only if $a = b$, the cycle from a to a can be of the length $n = 1$.

We'll explain later the important (and very fast) construction of the graph Γ_{R^∞}, which is known as the transitive closure of the initial graph Γ_R. This construction is called Warshall's Algorithm and is fundamental in practical computer applications. But before this we'll return to the definition of the composition $S \circ R$ and express this operation in the language of the Boolean matrices.

If M_R and M_S are Boolean matrices of the relations R and S, we can construct $M_{S\circ R}$ in the following way. Suppose that v_{ij} are the entries of M_R , S_{ij} are the entries of M_S and t_{ij} are the entries of $M_{S\circ R} = M_T$. Then, by the definition

$$t_{ij} \quad = \quad 1 \Leftrightarrow a_i \xrightarrow{T} a_j \Leftrightarrow \text{ there is element } a_k \text{ such that:}$$

$a_i \xrightarrow{R} a_k$ and $a_k \xrightarrow{S} a_j \Leftrightarrow$ there is index

$$k \quad : \quad v_{ik} = 1 \text{ and } S_{kj} = 1.$$

This operation over Boolean matrices is called the *Boolean product* $M_R \odot M_S$. Formally, the Boolean product $A \odot B$ of two Boolean $n \times n$ matrices A and B (with elements a_{ij} , b_{ij}) is the Boolean $n \times n$ matrix $C = [C_{ij}]$ defined by

$$C_{ij} = \begin{cases} 1 \ \text{ if } a_{iK} = 1 \ \text{ and } b_{Kj} = 1 \text{ for some } 1 \leq k \leq n \\ \qquad\qquad 0 \text{ otherwise.} \end{cases}$$

Why is it a product? Remember the definition of the product for two arbitrary $n \times n$ matrices $A = [a_{ij}],\ B = [b_{ij}]$. According to linear algebra,

$$\begin{aligned} C &= A \cdot B = [C_{ij}], \text{ where} \\ c_{ij} &= K = 1\sum^{n} a_{iK} \cdot b_{Kj}. \end{aligned}$$

That is, we have to multiply the elements of the i-th row of the matrix A (first factor) by corresponding elements of the j-th column of the matrix B (second factor) together, and then add the pairwise products to form C_{ij} :

$$\begin{bmatrix} a_{11}\ldots a_{1n} \\ \ldots\ldots\ldots\ldots \\ a_{i1}\ a_{iz}\ldots a_{in} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_{n1}\ldots\ldots a_{nn} \end{bmatrix} \begin{bmatrix} b_{11}\ldots b_{1j}\ldots b_{1n} \\ b_{21}\ldots b_{2j}\ldots b_{2n} \\ \\ \\ \\ b_{n1}\ldots b_{nj}\ldots b_{nn} \end{bmatrix}$$

$$\begin{matrix} a_{j1}............a_{in} \\ b_{1j}............b_{nj} \end{matrix} \quad \Rightarrow \quad C_{ij}.$$

If we construct the product of two Boolean matrices $C = AB$ (in the sense of linear algebra), the element $C_{ij} = K = 1\overset{n}{\sum} a_{ik}\ b_{kj}$ will be a non-negative integer, namely the number of $0 \leq k \leq n$ such that

$$a_{ik} = b_{kj} = 1.$$

Either $c_{ij} = 0$ and the same result gives the Boolean product $A \odot B$ (for the position (i, j)) or $C_{ij} \geq 1$ and then at the corresponding place in the Boolean product we'll get 1. In different words $\widetilde{C}_{ij} = k = 1\overset{n}{\bigvee} (a_{ik} \wedge b_{kj})$.

Modulo such technical details Boolean multiplication is similar to the standard matrix multiplication. It is associative

$$A \odot (B \odot C) = (A \odot B) \odot C$$

but as a rule noncommutative:

$$A \odot B \neq B \odot A$$

As we proved already (it is in fact the trivial consequence of the definitions)

$$M_{S \circ R} = M_R \odot M_S$$

In particular

$$M_{R^k} = K \text{ factors} \underbrace{M_R \odot M_R \odot ... \odot M_R} = M_R^{k\odot}$$

and

$$\begin{aligned} M_{R^\infty} &= M_{K=1\overset{n}{\cup} R^K} = M_R \vee M_{R^2} \vee ... \vee M_{R^n} = \\ &K = n\overset{n}{\vee} \left(M_R^{K\odot}\right) \end{aligned}$$

Effective construction of the connectivity relation R^∞ in terms of given relation R for small values of $n = |A|$ is rather easy. The best approach in this case is based on the graph representation and direct geometrical analysis. Algebraic calculations (Boolean products and joins) are also not very complicated.

Example. Relation R given by the following graph Γ_R or matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the structure of Γ_{R^∞} or M_{R^∞}?

Geometrical approach. From the state 1 we have path to vertices 1 and 2, it is impossible to reach 3 and 4. From 2 there is path to 1 and 2 (Set (1,2) has "absorbing" property). From 3 one can reach 1, 2, 4 but not 3. At least 4 is "absorbing" vertex. It means that

$$M_{R^\infty} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic manipulations are longer but more systematic. They can be used for computers (that is absolutely impossible for geometrical approach based on the visualization and intuition).

We have

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; M_R^{2\odot} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_R^{3\odot} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M_R \,, M_R^{4\odot} = M_R^{2\odot}$$

and

$$M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} = M_R \vee M_{R^2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Unfortunately, for large values of n computations by the formula

$$M_{R^\infty} = K = 1\overset{n}{V} M_{R^n} = K = 1\overset{n}{V} (M_R)^{n\odot}$$

are very long. As is easy to see, the Boolean multiplication of two $n \times n$ matrices requires $2n^3$ logical operations (n rows of $A \times n$ columns of $B \times 2n$ operation to "multiply" row and column by the formula $K = 1\overset{n}{V}(a_{iK} \wedge b_{Kj})$.

It means that evaluation of the matrix M_{R^∞} requires the number of operations of order $2n^4$ (even more!).

If $n = 10^3$ and computer has a speed 10^6 operations/sec time of evaluation of the M_{R^∞} will have the order $2 \cdot 10^6 \,\text{sec} = 25$ days!

The Warshall's algorithm (see sec 5.4 in the text-book) requires only $C \cdot n^3$ operations and time of calculations for the same value $n = 10^3$ will have order only 10 min!

Now I'll give a brief description of the Warshall algorithm.

Let Γ_R be a graph of relation R on a set $A = (a_1 ..., a_n)\,, |A| = n$. If $a \to x_1 \to x_2 \to ... x_{K-1} \to b$ is a path between a, b then vertices $x_1, ... x_{K-1}$ are called interior vertices of the path. Let's define a Boolean matrices $W_K, k = 0, 1, ..., n$ as follows:

$W_K = \left\{W_{ij}^{(K)}\right\}$, $W_{ij}^{(K)} = 1 \Leftrightarrow$ there is a path from a_i to a_j whose interior vertices, if any, come from the set $A_K = (a_1, ..., a_k)$. Endpoints a_i, a_j are arbitrary.

It is clear that $A_0 = \emptyset$ and $W_0 = M_R$. On the other hand, $A_n = A$ and $W_n = M_{R^\infty}$. We'll give simple rule how to evaluate W_k in terms of W_{k-1}. Starting from $W_0 = M_R$ and applying this rule n times, we'll construct M_{R^∞}. Suppose that $W_k = [W_{ij}^{(k)}]$, $W_{k-1} = [W_{ij}^{(k-1)}]$ since $A_{K-1} \subset A_K$, all the 1's from W_{k-1} go to W_k : if $W_{ij}^{(k-1)} = 1 \Rightarrow W_{ij}^{(k)} = 1$. Let $W_{ij}^{(k)} = 1$ but $W_{ij}^{(k)} = 0$. It means that there is a path from a_i to a_j with interior points from $(a_1, ..., a_k) = A_k$ including a_k! Without loss of generality (see theorem above), we can assume that all interior vertices are distinct. In this case our path has a form:

The subpath from i to k has the interior vertices from A_{k-1}, as well as subpath from k to j. In different terms

$$W_{ik}^{(k-1)} = 1, \; W_{kj}^{(k-1)} = 1.$$

(i is from the K-th column and j from the K-th row)

Now we have a following rule for transforming W_{k-1} into W_k :

Step 1. Transfer all 1's from W_{k-1} to W_k

Step 2. Take in W_{k-1} the k-th column and k-th row and list the locations $P_1, P_2, ... q_1, q_2 ...$ of the 1's in the k-th column and the locations of 1's in the k-th row.

Step 3. Put 1's in all positions (P_i, q_j) of W_k (if they are not already there).

Homework. Graph Γ is connected (by the definition) if $M_{\Gamma^\infty} = E = [e_{ij} \equiv 1]$.

a) Prove that number of edges in the connected graph with n vertices can not be less than n.

b) Prove that result a) can not be improved, i.e. construct connected graph F with n vertices and n edges.

c) Construct graph Γ with n vertices, $(2n-2)$ edges such that

$$M_\Gamma \vee M_{\Gamma^2} = E$$

Result of the problem c) gives an explanation of the 1 structure of the aircraft connection for such big companies as "Delta", "AA", etc.

Example 1. Graph Γ is shown on the fig. 1.

Find $\Gamma\infty$ using Warshall algorithm

Solution. As easy to see$M_\Gamma = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

Let's put $W_0 = M_\Gamma$. To construct W_1 let's take the first column and the first row of $W_0 = M_\Gamma$. 1's are located at the places with indices 1 and 2 in the first column, i.e. $P_1 = 1, P_2 = 2$. Corresponding locations of 1's in the first row are also $q_1 = 1, q_2 = 2$. We can construct four pairs $(P_i, q_j) : (1,1), (1,2), (2,1), (2,2)$ and put 1's only in position (2,2) (all others are already occupied). It means, that

$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$. Second column and the second row of W_1 give $P_1 =$

1, $P_2 = 2; q_1 = 1, q_2 = 2, q_3 = 4$ and it gives again only one new pair (i.e...2's) in the positions (1,4). It means that

$W_2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$. Third column and row of W_2 give $P_1 = 5, q_1 = 4$ i.e. pair (5,4).

In different words

$W_3 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. Corresponding fourth column and row give $P_1 =$ 1, $P_2 = 2$, $P_3 = 3$, $P_4 = 5$, $q_1 = 5$i.e. three new 1's with position (1,5), (2,5), (3,5). I.e.

$W_4 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. At the last step (fifth column and row) we have $P_1 = 1, P_2 = 2, P_3 = 4, P_4 = 4, P_5 = 5, q_1 = 3, q_2 = 4, q_3 = 5$ and many new pairs: (1,3), (2,3), (3,3), (4,3), (4,4). Final answer is

$$W_5 = M_{\Gamma\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Of course this answer is also clear geometrically.