Lectures in Discrete Mathematics

Math 1165

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Lecture 0. Introduction

Discrete mathematics continues to be an important part of mathematics education at the college level, especially since it plays such an important role in computer science. Our goal is to introduce several fundamental notions and tools: binary and other numerical systems, formal and mathematical logic and set theory, combinatorics, mathematical induction, many specific examples of the algorithms, graphs and relations, and Boolean functions and matrices. These notions are key components not only in the theoretical computer science, but also in practical computing.

Discrete mathematics in some sense is much older then the calculus or algebra. The simplest and most fundamental objects in discrete mathematics are the integers, which were known in the stone age (reference?). Formal logic, which includes rules for the reasoning, were developed by Plato and Aristotle in ancient Greece. Calculus, on the other hand, was not developed until Newton and Liebnitz in the 17th century. Thus, discrete math is roughly two thousand years older.

Discrete mathematics is not concerned the notions of modern (analytical) mathematics such as graphs of functions, coordinate systems, velocity, acceleration, and vectors but instead with much simpler objects like the positive integers and the theory of counting of finite sets of objects.

Let's illustrate some of the basic ideas of our subject by the examples. Although our examples can be understandable by bright pre-college students, they are not trivial. Such problems are typical of problems found in mathematical competitions.

Example 1. Sam and Tom play the following game. Sam thinks of an integer n between 1 and 1000, $1 \leq n \leq 1000$ and Tom has to guess it asking questions. The possible answers must be only "yes" and "no". Tom can ask, for instance: is your number divisible by 3, etc. What is an optimal strategy for Tom? What is a minimal number of questions that will guarantee that Tom will find the number?

Solution. The minimal number of questions is equal to 10 and the optimal strategy consists of a sequence of questions that has the effect of dividing the set of possible numbers in half (or as close to half as possible). For instance:

1. Tom: is $n > 500$? Sam: yes.
   Thus $500 < n \leq 1000$.

2. Tom: is $n > 750$? Sam: no.
   Thus $500 < n \leq 750$.

3. Tom: is $n > 625$? Sam: yes.
   Thus $625 < n \leq 750$.

4. Tom: is $n > 687$? $\left(687 \simeq \frac{750+625}{2}\right)$ Sam: no.
   Thus $625 < n \leq 687$.

Tom has succeeded in reducing the set of possibilities to 61. He can continue to reduce this number to $31, 16, 8, 4, 2$, and finally to 1, in at most six more guesses. Also, note that for any game like this that starts with a number in the range $1 \leq n \leq 1000$, following each guess, the maximum number of possibilities left is $500, 250, 125, 63, 32, 16, 8, 4, 2, 1$. Prove that 9 questions, generally speaking, are not sufficient. This example is closely related to the binary system, which we discuss in chapter 2.

Example 2. We are given a group of the 25 coins, 24 of which are normal and 1 which is lighter. We have to detect it, using a two-pan balance. The balance consist of a beam supported in the center and two identical pans on the left and right arms of the balance.

fig. 1

What is the minimal number of the weighings and what is the optimal strategy?

Solution. Our solution is based on the idea that we get optimal information about the identity of the light coin when we divide the coins into three (nearly) equal parts. Therefore, for the first weighing, let's put eight coins on each tray. Put first two groups on the pans (see fig. 2a, 2b).

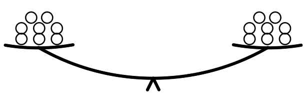

fig. 2a

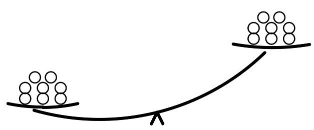

fig. 2b

In the case 2a the false coin is among the 9 other coins and in the case 2b it is one of the 8 on the right pan. On the second step, partition the 9 or 8 coins as follows: $9 = 3 + 3 + 3$ (case 2a) or $8 = 2 + 3 + 3$. At the third step we will detect light coin. Can you show that two weighings are insufficient?

Problem 1. Given a group of 29 coins that contains 2 false coins, both lighter than the others. What is your strategy for detecting these coins? Give an explanation.

Problem 2. Given a group of 81 coins that contains one false coin whose weight is distinct from the normal ones. Show that 5 weighings is sufficient to detect this coin and to decide whether the odd coin is lighter or heavier.

Let's give a brief solution of this problem based on the further development of the ideas of the Example 2.

Solution. Let's divide all coins into three equal groups, A, B, and C containing 27 coins each. Put A on the left pan and B on the right pan of the balance. There are three possibilities:

1. Equilibrium. Then the false coin is in group C. This group contains 27 coins and we can continue the same procedure: partition into 3 equal parts and select the group that contains the odd coin.

2. The set of coins in A is heavier. For the second weighing, replace A with C. The result of this will tell us which of the three sets has the odd coin and whether it is heavier or lighter.

3. The set of coins in A is heavier.

(1)

Of course, the false coin either on the left pan (than it is lighter) or on the right pan (that it is heavier). Remaining 27 coins are regular.

Let interchange groups (1) and (3). After this the balance can be in one of three following positions (see fig 4_1-4_3):

(2)

Equilibrium. In this case the coin is inside (1) and it is lighter ($m_f < m_r$). (Why?).

(3)

Here the false coin belongs to the group (2) and $m_f > m_r$.

(4)

Here again the false coin is in (2) but $m_f < m_r$. Anyway we detected now (using

two weighings) the group of 27 coins, containing the false one and the inequality between m_f and m_r. Due to problem 2 remaining 3 attempt is sufficient to find a false coin.

Problem 3. Group of 12 coins contains 1 false coin ($m_f \neq m_{reg}$!). Proof that 3 weighings are sufficient to detect it.

Example 3. "How many children do you have and how old are they" asked the guest, a mathematics teacher. "I have three boys" said Mr. Smith. "The product of their ages is 72 and the sum of their ages is the street number of our house." The guest went to look at the entrance, came back and said: "The problem is indeterminate." "Yes, that is so" said Mr. Smith "but I still hope that the oldest boy will some day compete in the UNC Charlotte mathematical competition." Tell the ages of the boys, stating your reasons.

Solution. Let $n_1 \leq n_2 \leq n_3$ be the ages of boys. We know that $n_1 n_2 n_3 = 72$. The fundamental fact (typical for the discrete mathematics!) is that we have only a finite number of possibilities for n_1, n_2, n_3. The next table contains all of them.

n_1	n_2	n_3	$n_1+n_2+n_3$
1	1	72	74
1	2	36	39
1	3	24	28
1	4	18	23
1	6	12	19
1	8	9	18
2	2	18	22
2	3	12	17
2	4	9	15
2	6	6	14
3	3	8	14
3	4	6	13

The only indeterminate case occurs when $S = n_1 + n_2 + n_3 = 14$. The last sentence of Mr. Smith contains the information about the oldest boy (not boys!). Therefore, the answer is $n_1 = n_2 = 3$ and $n_3 = 8$.

Example 4. The following example illustrate the ideas of the formal logic. It doesn't contain numbers at all. Three wise men dispute who is the smartest. They asked a judge to resolve the issue. "Look," said the judge, "I have five hats: three white and two red. Close your eyes, please." The judge put on each of the men the hat and said: "Now open your eyes, please. What is the color of your hat?" Each man could see the hats of the other two men but not his own.

After a long silence, one of the men said: "My hat is white." What was his reasoning?

<u>Solution</u> Assume that my (1) hat is red. Then my right neighbor (2) will see against him a white and a red hat. He will argue in the following way: "If my hat is red then (3) will see two red hats and immediately understand that his hat is white! But he did not say so, i.e., my hat is white!"

"However (2) did not announce his decision, it means that my hat is not red (by initial assumption) but a white hat."

This problem is closely related to the formal logic, its solution as well as solutions of many similar problems can be derived from the special system of the logical equations.