

1. Some properties of relations. We describe below some important properties that relations might or might not have. A relation  $R$  on a set  $A$  is called

R. Reflexive if  $\forall x \in A, xRx$ .

S. Symmetric if  $\forall x, y \in A, xRy \Rightarrow yRx$ .

A. Antisymmetric if  $\forall x, y \in A, xRy$  and  $yRx \Rightarrow x = y$ .

T. Transitive if  $\forall x, y, z \in A, xRy$  and  $yRz \Rightarrow xRz$ .

Now let  $A = \{1, 2, 3\}$ . There are  $2^4 = 16$  subsets of  $\{R, S, A, T\}$ . Find a relation on  $A$  for each of these subsets. For example, consider the subset  $\{S, A, T\}$ . We seek to find a relation on  $\{1, 2, 3\}$  that is symmetric, antisymmetric, and transitive, and *not* reflexive. To keep the relation from being reflexive, we must exclude one of the three ordered pairs  $(1, 1), (2, 2), (3, 3)$ . However, two of these could be included. So let's try  $H = \{(1, 1), (2, 2)\}$ . Is this symmetric? Is it transitive? Is it antisymmetric? Sketch the digraph of the relation and notice that it has just two loops. After some thought, you'll decide that  $H$  is symmetric, antisymmetric, and transitive. There are 15 other subsets of  $\{R, S, A, T\}$ . Find a relation for each of these, or prove that certain combinations do not exist.

2. For each relation (in the plane) defined below, list the properties that it satisfies. For example, the first relation is a circle centered at the origin, so it is symmetric, but it is not reflexive, antisymmetric, or transitive.

(a)  $x^2 + y^2 - 1 = 0$

(b)  $xy(y - x)(y + x) = 0$

(c)  $(x - \lfloor x \rfloor)(y - \lfloor y \rfloor) = 0$

(d)  $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 1$

(e)  $\lfloor x^2 \rfloor + \lfloor y^2 \rfloor = 1$

3. Let  $R$  and  $T$  be the relations on  $A = \{1, 2, 3, 4, 5, 6\}$  defined as follows:

$$aRb \iff a - b \text{ is divisible by } 2$$

$$aTb \iff a \leq b$$

Construct the Boolean matrices and the digraph for each of the relations  $R, T, \bar{T}, R^{-1}, R \cup T, R \cap T$  and determine which of the properties listed in problem 1 are satisfied by each of these. See lecture 9 for the definitions of  $\bar{R}$  and  $R^{-1}$ .

4. Let  $N$  be an  $n$  element set. Then there are  $2^{n^2}$  relations on  $N$ . Thus there are  $2^9 = 512$  relations on a three element set. Use the matrix model or the digraph model to find the number of relations on the set  $S = \{1, 2, 3\}$  that are
- (a) reflexive (R)
  - (b) symmetric (S)
  - (c) antisymmetric (A)
  - (d) transitive (T)
  - (e) equivalence relations (RST)
  - (f) partial orderings (RAT)
5. Define a relation  $R$  on  $Z$  as follows:  $aRb \Leftrightarrow b - a$  is a multiple of 7. This is another way of saying  $aRb \Leftrightarrow b \equiv a \pmod{7}$ . Prove that  $R$  is an equivalence relation on  $Z$ . What are the cells of the partition determined by  $R$ ? Define the sum and the product of cells and construct the two arithmetic tables.
6. A positive six-digit integer  $d_1d_2 \dots d_6$ , where  $d_1 \neq 0$  is said to be *3-special* if each of the numbers  $10d_1 + d_2, 10d_2 + d_3, \dots, 10d_5 + d_6$  is a multiple of 3. How many 3-special numbers there?