Throughout we use both the notations $\binom{n}{r}$ and C_r^n for the number $\frac{n!}{(n-r)!r!}$.

1. Ten points are distributed around a circle. How many triangles have all three of their vertices in this 10-element set?

2. Let $\mathcal{U} = \{0,1,2,3,4,5,6,7,8,9\}$ be the universal set. Let $S = \{1,2,3,4,5\}$ and $T = \{4,5,6,7,8\}$.

   (a) How many four-element subsets A of $\mathcal{U}$ satisfy $|A \cap S| = 2$ and $|A \cap T| = 2$?

   (b) Let D denote the set of all four-digit numbers that can be built using the elements of S as digits and allowing repetition of digits. What is $|D|$?

   (c) How many elements of D have four different digits?

   (d) How many elements of D have exactly three different digits?

   (e) How many even numbers belong to D?

3. The following problems are related.

   (a) What is the value of $\frac{7!}{(7-3)!3!}$?

   (b) How many 3-element subsets does the set $\{A, B, C, D, E, F, G\}$ have?

   (c) How many solutions are there to

   $$x + y + u + v = 4$$

   where x, y, u, and v are nonnegative integers. For example, $(2, 1, 0, 1)$ is such a solution.

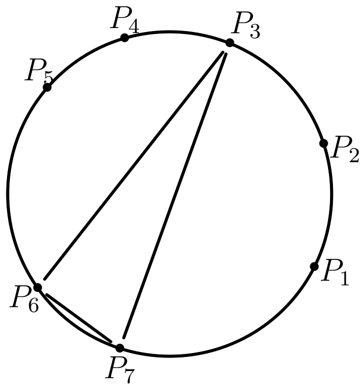
   (d) How many solutions does

   $$x + y + u + v = 8$$

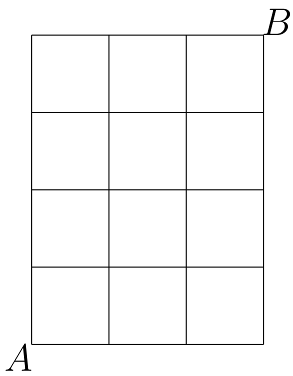
   have subject to the condition that each of the variables is a positive integer?

   (e) How many ways can a 3-person committee be selected from a 7-member club?

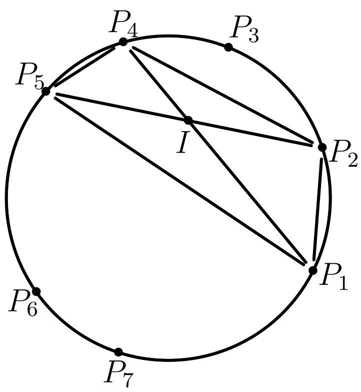
(f) Let $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ be seven points distributed around a circle. How many triangles have all three vertices in the set.



(g) How many paths of length 7 are there from A to B in the grid below?



(h) Seven points are distributed around a circle. All pairs of them are joined by a secant line. What is the largest possible number of points of intersection *inside* the circle?



(i) What is the coefficient of x^3 in the expanded form of $(x+1)^7$?

(j) What is the third entry of the seventh row of Pascal's triangle?

(k) How many numbers can be expressed as a sum of four distinct members of the set $\{1, 2, 4, 8, 16, 32, 64\}$?

4. A *falling* number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example 96521 is a falling number but 89642 is not. How many n-digit falling numbers are there, for $n = 1, 2, 3, 4, 5, 6, 7, 8,$ and 9? What is the total number of falling numbers of all sizes?

5. Cyprian writes down the middle number in each of the $\binom{9}{5} = 126$ five-element subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then he adds all these numbers together. What sum does he get?

6. Counting sums of subset members.

   (a) How many number can be expressed as a sum of two or more distinct elements of the set $\{1, -3, 9, -27, 81, -243\}$?

   (b) How many numbers can be expressed as a sum of two or more distinct members of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

   (c) How many numbers can be expressed as a sum of four distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

   (d) How many numbers can be expressed as a sum of two or more distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

7. How many positive integers less than 400 have exactly 6 positive integer divisors?

8. How many of the first 242 positive integers are expressible as a sum of three or fewer members of the set $\{3^0, 3^1, 3^2, 3^3, 3^4\}$ if we are allowed to use the same power more than once. For example, $5 = 3 + 1 + 1$ can be represented, but 8 cannot. Hint: think about the ternary representations.

9. How many integers can be expressed as a sum of two or more different members of the set $\{0, 1, 2, 4, 8, 16, 32\}$?

10. John has 2 pennies, 3 nickels, 2 dimes, 3 quarters, and 8 dollars. For how many different amounts can John make an exact purchase (with no change required)?

11. How many paths consisting of a sequence of horizontal and/or vertical line segments with each segment connecting a pair of adjacent letters in the diagram below, is the word **CONTEST** spelled out as the path is traversed from beginning to end?

C
C O C
C O N O C
C O N T N O C
C O N T E T N O C
C O N T E S E T N O C
C O N T E S T S E T N O C

12. Recall that a Yahtzee Roll is a roll of five indistinguishable dice.
a. How many different Yahtzee Rolls are possible? b. How many Yahtzee Rolls have exactly 3 different numbers showing?

13. How many four digit numbers $\underline{abcd}$ satisfy $|a - d| = 2$?

14. How many numbers can be expressed as a sum of three distinct members of the set $\{4, 5, 6, 7, 8, 9, 10, 11, 12\}$?

15. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
a. How many five element subsets does the set have?
b. How many subsets of S have an odd number of members?
c. How many subsets of S have 1 as a member?
d. How many subsets have 1 as a member and do not have 2 as a member?

16. Imagine that the 4×7 grid of squares below represents the streets of a part of the city where you live. You must walk 11 blocks to get from the lower left corner at A to the upper right corner at B.

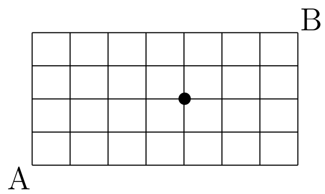
    (a) How many different 11 block walks are there?

    (b) How many 11 block walks avoid the terrible corner marked with the bullet?

    (c) How many 11 block walks go through the terrible corner?

    (d) How many different 12 block walks are there from A to B?

    (e) How many different 13 block walks are there from A to B?



17. How many positive integers less than 1000 have an odd number of positive integer divisors?

18. How many integers can be obtained as a sum of two or more of the numbers $1, 3, 5, 10, 20, 50, 82$?

19. How many four-digit numbers have the property that the sum of the first three digits is the fourth digit. For example 1247 has the property.

20. How many numbers in the set $\{100, 101, 102, \ldots, 999\}$ have a sum of digits equal to 9? B. How many four digit numbers have a sum of digits 9? C. How many integers less than one million have a sum of digits equal to 9? **August 13, 2003 2:18 P.M.**