Homework 5B, The Pigeonhole Principle

1. Prove that for each integer n there exists a pair of different positive integers a and b such that $n^a - n^b$ is a multiple of 10. What if we change 10 to 100?

2. Prove that any 68 element subset S of $\{1, 2, 3, \ldots, 100\}$ must contain three consecutive integers.

3. Is it true that from any 30 different natural numbers, not greater than 50 one can choose a pair such that one number in the pair is twice the other.

4. There are 33 students in the class and sum of their ages 430 years. Is it true that one can find 20 students in the class such that sum of their ages greater 260?

5. The number $7 \cdot 5^{41}$ is a 30 digit number. Prove that some digit appears at least 4 times in the decimal representation.

6. Prove that every set of five vertices of a cube contains three points that are the vertices of an equilateral triangle.

7. Prove that 7 has a multiple of the form $111\ldots1$ (called *a repunit*). Is the same proposition true for 19? Now prove that for any n, n has a repunit multiple.

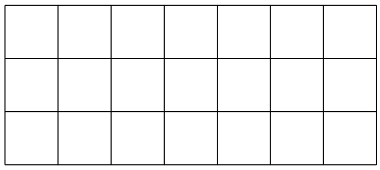
8. Show that every eleven element subset of $S = \{1, 2, 3, 4, \ldots, 62\}$ contains three numbers which are the lengths of the legs of some triangle.

9. Twelve members of a club agree to send cards to one another subject to the following conditions: each person will randomly select six others to send cards to. (Hint: there are only 66 pairs of persons in the club.)

   i. Prove that some pair of persons exchange cards.

   ii. Does the same conclusion hold if each member of the club agrees to send five cards.

10. If each square of a 3-by-7 chessboard is colored either black or white, then the board must contain a rectangle consisting of at least four squares whose corner squares are either all white or all black.



Show that if the grid is only 3-by-6, there are colorings for which the conclusion fails.

11. Fifty-one points are scattered inside a square with a side of one meter. Prove that some subset of three of these points can be covered by a square with side 20 centimeters.

12. Fifteen children together gathered 100 nuts. Prove that some pair of children gathered the same number of nuts.

13. The integers from 1 to 10 are randomly distributed around a circle. Prove that there must be three neighbors whose sum is at least 17. What about 18? What about 19?

14. The digits $1, 2, 3, \cdots, 9$ are divided up into three groups. Prove that the product of the numbers in one of the groups must exceed 71.

15. A set of 1998 different positive integers is given. None of these numbers can be represented as a sum of two others from the set. What is the least possible value for the largest number from such a set?

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