Homework 5, Solving Recurrence Equations Math 1165,

1. Recursion, Closed Form, etc.

   (a) Consider the sequence $1/1, 1/3, 1/5, 1/7, 1/9 \ldots$. Please note that sequences like the one above are strictly speaking, not well defined, because there is no way to be sure that the reader has in mind the same sequence the author has in mind. However, I have decided to take the risk.

      i. Find a closed form (i.e. general form) formula.

      ii. Find a first order recursive definition of the sequence.

   (b) Consider the sequence $1, 4, 9, 16, 25, \ldots$.

      i. Find a closed form formula.

      ii. Find a first order recursive definition of the sequence.

   (c) Consider the sequence $0, 4, 16, 36, 64, 100 \ldots$.

      i. Find a closed form formula.

      ii. Find a first order recursive definition of the sequence.

   (d) Consider the sequence $1, 3/2, 11/6, 50/24, 274/120 \ldots$.

      i. Find a closed form formula.

      ii. Find a first order recursive definition of the sequence.

   (e) Consider the sequence $a_1 = \sqrt{2}, a_2 = \sqrt{2 + \sqrt{2}}, a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$.

      i. Show that there exists a number M for which $a_n < M$ for all positive integers n. This is another way to say the sequence is bounded above.

      ii. Show that $a_{n-1} < a_n$ for all integers $n \geq 2$.

      iii. Find a closed form formula.

      iv. Find a recursive definition of the sequence.

      v. Find the limit of the sequence, if it exists.

   (f) Consider the sequence $0, 0.1, 0.11, 0.111, 0.1111, \ldots$.

      i. Find a closed form formula.

      ii. Find a recursive definition of the sequence.

   (g) Consider the sequence $0, 1, 3, 6, 10, 15, 21, \ldots$.

      i. Find a closed form formula.

      ii. Find a recursive definition of the sequence.

2. The Lucas numbers are defined by $L_0 = 2, L_1 = 1,$ and $L_{n+2} = L_{n+1} + L_n$. Find the characteristic equation for the recursion, solve it, and then find the unique general form for the Lucas numbers. Use the general form to find L_{100}.

3. Consider the recursively defined sequence $x_1 = 3, x_2 = 5,$ and for $n > 2$, $x_n = \frac{x_{n-1}+1}{x_{n-2}}$.

   (a) Find the first 10 terms.

   (b) Prove that for any initial values, the sequence is periodic with period 5.

4. Use the method discussed in the notes and in class to find a formula for the n^{th} term of the sequence defined by $a_1 = -1, a_2 = 1,$ and $a_{n+2} = 5a_{n+1} - 6a_n$. Then use mathematical induction to prove your answer.

5. The case of repeated roots. When the characteristic equation has repeated roots, say $x = \lambda, x = \lambda$, try the two solutions $a_n = \lambda^n$ and $a_n = n\lambda^n$. Use this idea to solve $a_1 = 0, a_2 = 4,$ and $a_{n+2} = 4a_{n+1} - 4a_n$. Then prove that your formula works.

6. Turning the method around. Find a recurrence relation that has the solution $a_n = 2^{n+2} + n \cdot 2^{n+1} + n^2 \cdot 2^n$.

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