Homework 1 Math 1165

Number Theory

June 4, 2003 8:43 A.M.

1. For each of the fractions listed, find the decimal equivalent.
   (a) 1/7
   (b) 1/27
   (c) 1/81.
   (d) 1/243

2. Explain why every ratio of integers either terminates with zeros from some point on or repeats in blocks from some point on.

3. For each of the repeating decimals below, find the decimal ratio of integers.
   (a) $0.\bar{4}$
   (b) $1.2\bar{4}$
   (c) $2.34\overline{45}$
   (d) $0.\bar{9}$

4. Properties of divisibility. Prove each of the following properties. Read Lecture 1 before you try these.
   (a) For each integer d, $d|d$ (reflexivity).
   (b) If $d|n$ and $d|m$, then $d|\,(n+m)$ (additivity). In other words, the sum and the difference of two multiples of d is a multiple of d.
   (c) If a and b are positive integers and $a|b$ and $b|a$, then $a = b$ (antisymmetry).
   (d) If $d|n$ and $n|l$, then $d|l$ (transitivity). In other words, of multiple of a multiple of d is a multiple of d.

5. Divisibility Foundation for the Euclidean Algorithm. Suppose m and n are given positive integers, with $n < m$. The division of m by n results a unique pair of numbers, a quotient q and a remainder r satisfying $m = q \cdot n + r$ and $0 \le r < n$.
   (a) Prove that if $d|m$ and $d|n$, then $d|r$.

(b) Prove that if $d|r$ and $d|n$, then $d|m$.

This proves that the set of common divisors of m and n is the same as the set of common divisors of n and r. The problem of finding the gcd of two integers is reduced to an easier problem of the same type. Thus the greatest common divisor of m and n is the same as that of d and n.

6. Carry out the division algorithm for each of the pairs below. Read Lecture 1 before you try these problems. In particular note from the lecture that the remainder r cannot be negative.

   (a) $n = 77$ and $d = 17$.

   (b) $n = -77$ and $d = 17$.

   (c) $n = 1999$ and $d = 77$.

   (d) $n = 77$ and $d = -17$.

7. Sketch the graphs of these two functions

$$h(x) = \lfloor x \rfloor \text{ and } f(x) = \{x\} \text{ for } -\infty < x < \infty.$$

8. When is n divisible by 3? How can you tell the remainder when n is divided by 3 quickly from the digits of n?

9. Let a, b, and c be decimal digits. Prove that the six-digit number $n = abc, abc$ is divisible by 7 and 13. Hint: $1001 = 7 \cdot 11 \cdot 13$.

10. Consider the following numbers

$$N_r = p_1 \cdot p_2 \cdot \ldots \cdot p_r + 1$$

If $r = 1$, $N_1 = 2 + 1 = 3$, which is prime. If $r = 2$, $N_2 = 2 \cdot 3 + 1 = 7$, which is also prime. If $r = 3$, $N_3 = 2 \cdot 3 \cdot 5 + 1 = 31$, which is prime as well. Using the table for primes ($2 \le p < 100$) prove that N_4, N_5 are prime, but N_6 is not prime.

11. Find prime number decompositions for

$$n = 1234, \quad m = 34560$$

and evaluate GCD(n, m), LCM(n, m).

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