Let

$$f(x) = \begin{cases} 2x+1 & \text{if } x < -2 \\ 3x-2 & \text{if } x \geq -2 \end{cases},$$

let

$$g(x) = \begin{cases} |x+3| & \text{if } x \leq 0 \\ x^2-4 & \text{if } x > 0 \end{cases},$$

and let $h(x) = 2x - 3$. Compute each of the following composite functions.

1. $f \circ h(x)$

   Solution: $f \circ h(x) = \begin{cases} 4x-5 & \text{if } x < 1/2 \\ 6x-11 & \text{if } x \geq 1/2 \end{cases}.$

2. $h \circ f(x)$

   Solution: $h \circ f(x) = \begin{cases} 4x-1 & \text{if } x < -2 \\ 6x-7 & \text{if } x \geq -2 \end{cases}.$

3. $h \circ g(x)$

   Solution: $h \circ g(x) = \begin{cases} 2|x+3|-3 & \text{if } x \leq 0 \\ 2x^2-11 & \text{if } x > 0 \end{cases}.$

4. $g \circ h(x)$

   Solution: $g \circ h(x) = \begin{cases} |2x| & \text{if } x \leq 3/2 \\ 4x^2-12x+5 & \text{if } x > 3/2 \end{cases}.$

5. $f \circ g(x)$

   Solution: The last two are much harder. For these, we need to successively refine the functions. $f \circ g(x) = \begin{cases} 2|x+3|+1 & \text{if } x \leq 0 \text{ and } |x+3| < -2 \\ 3|x+3|-2 & \text{if } x \leq 0 \text{ and } |x+3| \geq -2 \\ 2(x^2-4)+1 & \text{if } x > 0 \text{ and } x^2-4 < -2 \\ 3(x^2-4)-2 & \text{if } x > 0 \text{ and } x^2-4 \geq -2 \end{cases},$ which can be simplified to $f \circ g(x) = \begin{cases} 3|x+3|-2 & \text{if } x \leq 0 \\ 2x^2-7 & \text{if } 0 < x < \sqrt{2} \\ 3x^2-14 & \text{if } \sqrt{2} \leq x \end{cases}$ because the inequality $|x+3| < -2$ is not true for any real numbers.

6. $g \circ f(x)$

Solution: $g \circ f(x) = \begin{cases} |2x+1+3| & \text{if } x < -2 \text{ and } 2x+1 \leq 0 \\ (2x+1)^2 - 4 & \text{if } x < -2 \text{ and } 2x+1 > 0 \\ |3x-2+3| & \text{if } x \geq -2 \text{ and } 3x-2 \leq 0 \\ (3x-2)^2 - 4 & \text{if } x \geq -2 \text{ and } 3x-2 > 0 \end{cases}$, which simplifies to $g \circ f(x) = \begin{cases} |2x+4| & \text{if } x < -2 \\ |3x+1| & \text{if } -2 \leq x < 2/3 \\ 9x^2 - 12x & \text{if } x > 2/3 \end{cases}$.