Problem List

MATH 5173 Spring, 2009

The notation p/n means the problem with number $n$ on page $p$ of Perko.

1. 5/3
2. 6/5 and describe the relationship of the phase portraits
3. 6/6
4. 9/1
5. 9/3a
6. 10/5
7. Suppose $\dot{x} = \lambda x$ and $\dot{y} = \mu y$ for $\lambda, \mu > 0$. Obtain a formula of the form $x = h(y)$, $y = h(x)$, or $h(x, y) = 0$ for the solution curve through $(x_0, y_0) \neq (0, 0)$, for all such pairs $(x_0, y_0)$. Draw all qualitatively distinct phase portraits (there are three of them).
8. 19/6
9. 19/8
10. For $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ show that $\exp(tJ) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$ by writing
$$\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} \lambda t & 0 \\ 0 & \lambda t \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} := S + N$$
and computing $\exp(N)$ directly from the definition.
11. For $x(t) = x_0 + y_0 te^{\lambda t}$ and $y(t) = y_0 e^{\lambda t}$, the solution to the linear system of the previous problem, prove that if $\lambda y_0 \neq 0$ then $x = \frac{x_0}{y_0} y + \frac{1}{\lambda} \log \frac{y}{y_0}$.
12. Show that in polar coordinates the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ (for any functions $P$ and $Q$, not necessarily linear) becomes
$$\begin{aligned} r^2\dot{\theta} &= [xQ(x, y) - yP(x, y)]|_{(x=r\cos\theta, y=r\sin\theta)} \\ r\dot{r} &= [xP(x, y) + yQ(x, y)]|_{(x=r\cos\theta, y=r\sin\theta)}. \end{aligned}$$
Hint. Exercise 27/9 of Perko.
13. For the system $\dot{\mathbf{x}} = J\mathbf{x}$, $J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $b \neq 0$, change to polar coordinates to obtain an uncoupled system $\dot{\theta} = f(\theta)$, $\dot{r} = g(r)$. Solve the system explicitly (find $\theta(t)$ and $r(t)$) and use the solution to rigorously derive the phase portraits possible (with attention to the signs of $a$ and $b \neq 0$).
14. 26/1 Note that you are not to solve the system.
15. For each saddle in 26/1 find the equations of the separatrices. (They are the eigenspaces.)
16. 26/5
17. Show that if $\eta(t)$ solves $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x}(t_0) = \mathbf{x}_0$, then there exists a constant $c$ such that $\mu(t) := \eta(t + c)$ solves $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Find $c$ explicitly.

18. Suppose $f : I \times E \subset \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and consider the initial value problem
$$\dot{\mathbf{y}} = f(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{1}$$
and the integral equation
$$\eta(t) = \mathbf{y}_0 + \int_{t_0}^{t} f(s, \eta(s))\, ds. \tag{2}$$
    a. Show that if $\eta : J \subset I \to E$ is continuous and satisfies (2) then $\eta$ is differentiable and satisfies (1).
    b. Show that if $\eta : J \subset I \to E$ is differentiable and satisfies (1) then the integral in (2) exists and $\eta$ satisfies (2).

19. Let $\mu(t)$ be any solution to the initial value problem $\dot{\mathbf{y}} = f(t, \mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$, on some open interval $J_1 \subset \mathbf{R}$ about $t_0$. Prove by induction that (shrinking $J_1$ if necessary)
$$||\mu(t) - \eta_j(t)|| \leq M \frac{K^j (t - t_0)^j}{j!}$$
for all $j \in \mathbf{N} \cup \{0\}$, where the $\eta_j$ are the functions defined in the proof of the existence/uniqueness theorem. Use this estimate to show that $\eta_j \to \mu$ (at least pointwise) on $J_1$, so that $\mu = \eta$ on $J_1$.

20. Prove the existence/uniqueness theorem using the Contraction Mapping Principle. (See 76/5 of Perko.)

21. In the proof of Lemma 1 on page 85, we assumed in a proof by contradiction that an endpoint $t_0$ of $I^*$ lay in the interval $I$. Prove that if this is so then there exists a neighborhood $N$ of $t_0$ in $I$ on which both $u_1$ and $u_2$ are defined and continuous, that both $\lim_{t \to t_0} u_1(t)$ and $\lim_{t \to t_0} u_1(t)$ exist and have a common value $\mathbf{u}_0$, and that $\mathbf{u}_0 \in E$.

22. 92/2

23. 92/4

24. Suppose $f : E \subset \mathbf{R}^n \to \mathbf{R}^n$ is at least $C^1$ and that $\varphi : (\alpha, \beta) \to \mathbf{R}^n$ is a solution of the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0 \in E$. Let $\mathbf{g}(\mathbf{x}) = k\mathbf{f}(\mathbf{x})$, $k \in \mathbf{R} \setminus \{0\}$. Express the solution $\eta(t)$ of the initial value problem $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$, in terms of $\varphi(t)$, being careful about its domain. Discuss the relationship of the phase portraits of the two differential equations on $E$. (Compare with Problem 2 on this list.)

25. Same problem as Problem 24 but for $\mathbf{g}(\mathbf{x}) = k(\mathbf{x})\mathbf{f}(\mathbf{x})$, where $k : \mathbf{R}^n \to \mathbf{R} \setminus \{0\}$ is at least $C^1$.

26. Prove that if $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{f}$ is at least $C^1$, and there exists $M \in \mathbf{R}$ such that $|\mathbf{f}(\mathbf{x})| \leq M$ for all $\mathbf{x} \in \mathbf{R}^n$, then for any $\mathbf{x}_0 \in \mathbf{R}^n$ the unique solution to the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$, exists for all $t \in \mathbf{R}$.

27. Prove that if $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ and is at least $C^1$, and if we are studying geometric properties of the phase portrait of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ of ordinary differential equations on $\mathbf{R}^n$, then we may assume without loss of generality that solutions exist on all of $\mathbf{R}$. Hint. You may use the results of the previous two problems.

28. 97/2

29. 98/7

30. Duffing's equation is $\ddot{x} - x + x^3 = 0$
    a. Write an equivalent system of first order equations
    (*) $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$.
    b. Show that $H : \mathbf{R}^2 \to \mathbf{R} : (x, y) \mapsto \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2}y^2$ is constant on trajectories, so that every level curve of $H$ is an invariant set for (*).
    c. Using a symbolic manipulator like Maple or Mathematica, or by hand, find $H^{-1}(h)$ for any value of $h$ for which it contains an equilibrium of (*), and for nearby $h$. Using (*), decompose these level curves into trajectories and add arrows.

31. In the proof of the Flowbox Theorem, $f(\mathbf{0}) = (k, 0, \dots, 0)^T$ and $g$ was defined by

$$\begin{aligned} g : [-\alpha, \alpha] \times N \subset \mathbf{R} \times \mathbf{R}^{n-1} &\equiv \mathbf{R}^n \to \mathbf{R}^n \\ : (t, (x_2, \dots, x_n)) &\mapsto \eta(t, (0, x_2, \dots, x_n)). \end{aligned}$$

    a. Compute $dg(0, \mathbf{0})$. The answer is an $n \times n$ matrix of specific constants. Hint. The first column is $\dot{\eta}$. For the remaining columns you may set $t = 0$ before computing the first partial derivatives.
    b. Apply the Inverse Function Theorem to conclude that there exists a neighborhood $V$ of $(0, \mathbf{0})$ in $[-\alpha, \alpha] \times N$ and a $C^r$ mapping $h : W := g(V) \to V$ such that (i) $g \circ h = id_U$ and (ii) $h \circ g = id_V$.

32. In the context of the previous problem define a family of mappings on $V$, indexed by $t \in [-\alpha, \alpha]$, by

$$\mu(t, (x_1, x_2, \dots, x_n)) = h(\eta(t, g(x_1, x_2, \dots, x_n))).$$

Use fact b.(ii) from the previous problem and the definition of $g$ to conclude that $\mu(t, (x_1, \dots, x_n)) = (x_1 + t, x_2, \dots, x_n)$.

Note that the displayed equation is precisely the statement that $\mu(t, \mathbf{x})$ and $\eta(t, \mathbf{x})$ are topologically conjugate, with conjugating homeomorphisms $h$ and its inverse $g$.

33. Let $L$ be a hyperbolic linear mapping on $\mathbf{R}^n$ and let $\varphi(t, \mathbf{x})$ denote the global flow $\varphi(t, \mathbf{x}) = \exp(tL)\mathbf{x}$ of $\dot{\mathbf{x}} = L\mathbf{x}$. Let $C^0_*(\mathbf{R}^n)$ denote the set of mappings from $\mathbf{R}^n$ to itself that are uniformly continuous and uniformly bounded, and for $\mu > 0$ let $\mathcal{L}_\mu$ denote the set of mappings of the form $\Lambda = L + \lambda$ where $\lambda \in C^0_*(\mathbf{R}^n)$ is uniformly bounded by $\mu$ and is Lipschitz with Lipschitz constant at most $\mu$. For $\Lambda \in \mathcal{L}_\mu$ let $\eta(t, \mathbf{x})$ denote the local flow generated by $\dot{\mathbf{x}} = \Lambda(\mathbf{x})$. Prove that if $\mu > 0$ is sufficiently small, then for each $\Lambda \in \mathcal{L}_\mu$, $\eta(t, \mathbf{x})$ is actually defined on $[0, \infty)$. Address the question of whether the uniform bound on $\lambda$ must be small, or if the control of the size of its Lipschitz constant suffices.
Hint. Let $T > 0$ be given. For any $t \in [0, T]$ for which $\eta(t, \mathbf{x})$ exists bound $|\eta(t, \mathbf{x}) - \mathbf{x}|$ above by some quantity involving $T$ (hence independent of $t$) using the triangle inequality vis-à-vis $\varphi(t, \mathbf{x})$ and an integral equation that $\eta(t, \mathbf{x})$ satisfies on its maximal interval of existence $I(\mathbf{x})$. Conclude that $\eta(t, \mathbf{x})$ exists on $[0, T]$.

34. Use the Hartman-Grobman Theorem to classify the singularity specified, or state why the theorem does not apply. In the case of saddle points, use the Stable

Manifold Theorem to identify the tangent lines at the singularity to the stable and unstable separatrices.
a. $\dot{x} = x(x-y)$, $\dot{y} = y(2x-y)$ at $(0,0)$.
b. $\dot{x} = x - x^3$, $\dot{y} = -2y + y^2$ at $(0,0)$.
c. $\dot{x} = x - y$, $\dot{y} = 1 - e^x$ at $(0,0)$.
d. $\dot{x} = x + e^{-y}$, $\dot{y} = -y$ at $(-1,0)$.
e. $\dot{x} = y + x - x^3$, $\dot{y} = -y$ at each singularity.

35. Classify the singularity at the origin of the system $\dot{x} = 2x^2 - y^2$, $\dot{y} = 3xy$ as either stable, asymptotically stable, or unstable. (The $x$-axis is invariant and each of the half-planes it determines is a global elliptic sector.)

36. Any two parts of 130/2

37. Any two parts of 132/5

38. 132/7 and 132/8.
    Hint. Recall Example 4 on page 130, and ignore the author's hint.

39. 202/1

40. 203/2

41. 204/4

42. 206/8 Note that he is saying that $\gamma_1(t)$ and $\gamma_2(t)$ actually solve the system of differential equations. Such knowledge of explicit solutions (including the parametrizations) is rare.

43. Suppose $\gamma$ is a periodic orbit of a smooth system $\dot{\mathbf{x}} = f(\mathbf{x})$ on an open subset of $\mathbf{R}^2$, that $\Sigma$ is a section of the flow at some point of $\gamma$, that $s$ is a coordinate on $\Sigma$ chosen so that $s = 0$ corresponds to a point of $\gamma$, that $P(s)$ is the Poincaré first return map on a neighborhood of 0, and that $d(s)$ is the difference map $d(s) = P(s) - s$. Describe the orbit structure in a neighborhood of $\gamma$ in terms of the first non-zero derivative $d^{(k)}(0)$ of $d$ at 0. Argue carefully in terms of analysis, but intuitively in terms of geometry.
    Hint. What is important is the parity of $k$ and the sign of $d^{(k)}(0)$.

44. Prove as a corollary to the Poincaré-Bendixson Theorem the Poincaré Annular Region Theorem:
    Suppose $\mathscr{A}$ is the diffeomorphic image of an annulus. Suppose $f$ is a smooth vector field defined on a neighborhood of $\mathscr{A}$ with the following properties:
    (a) $f$ points into $\mathscr{A}$ (respectively, out of $\mathscr{A}$) at every point of $\partial\mathscr{A}$;
    (b) $f$ has no critical point in $\mathscr{A} \cup \partial\mathscr{A}$.
    Then $f$ has a cycle that lies wholly within $\mathscr{A}$.
    (We will see later that the cycle will have to enclose the hole in $\mathscr{A}$ in its interior.)

45. Suppose a smooth vector field $f$ has two unstable cycles $\gamma_1$ and $\gamma_2$, one in the interior of the other, in its domain of definition. Show that if $f$ has no critical point in the annular region bounded by $\gamma_1$ and $\gamma_2$ then it contains at least one stable cycle in that region. (Of course the analogous statement regarding existence of an unstable cycle is true if $\gamma_1$ and $\gamma_2$ are stable cycles.)
    Hint. Use the following fact: if $\gamma$ is an unstable cycle then there exists an annular neighborhood $\mathscr{N}$ of $\gamma$ such that $f$ points outward at every point of $\partial\mathscr{N}$.

46. If in the situation of Problem 44 $\operatorname{div} f$ is not identically zero and is of one sign on $\mathscr{A}$ then the cycle in $\mathscr{A}$ is unique.
    Hint. Use Theorem 2 and its corollary in section 3.4 of Perko and Problem 45.

47. Prove Dulac's Criterion:
    Suppose $E \subset \mathbf{R}^2$ is a simply connected open set, $f, B : E \to \mathbf{R}^2$ are $C^r$, $r \geq 1$, and $\operatorname{div}(Bf)$ is not identically zero on $E$ and is of one sign on $E$. Then no closed orbit of the system $\dot{\mathbf{x}} = f(\mathbf{x})$ lies wholly within $E$. (The function $B$ is termed a "Dulac function.")
    Hint. Resist the urge to apply the Product Rule.

48. Show that the planar system

$$\begin{aligned} \dot{x} &= 2xy \\ \dot{y} &= 2xy - x^2 + y^2 + 1 \end{aligned}$$

    has no closed orbits.
    Hint. Figure out why no closed orbit can intersect the $y$-axis. Show that a Dulac function is $B(x, y) = 2/x^2$.

49. Show that the planar system

$$\begin{aligned} \dot{x} &= \phantom{-}x(1 + x^2 - 2y^2) \\ \dot{y} &= -y(1 - 4x^2 + 3y^2 \end{aligned}$$

    has no closed orbits by showing that there exist positive constants $r$ and $s$ such that $B(x, y) = x^{-r}y^{-s}$ is a Dulac function.

50. It can be shown (by constructing a suitable Poincaré Annular Region) that the van der Pol oscillator

$$\begin{aligned} \dot{x} &= \phantom{-}y \\ \dot{y} &= -x + \lambda(1 - x^2)y \end{aligned}$$

    (studied by Balthasar van der Pol in the 1920's) has at least one limit cycle for all $\lambda > 0$. Use the Dulac function $B(x, y) = (x^2 + y^2 - 1)^{-\frac{1}{2}}$ (discovered by Leonid Cherkas in the late 20th century) to show that the cycle is unique.
    Hint. The function $B$ fails to exist on the unit circle.

51. 305/4 and 306/7