TWO TO ONE IMAGES AND PFA

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ABSTRACT. We prove that all maps on N^* that are exactly two to one, are trivial if PFA is assumed.

1. Introduction

A map $f : X \to K$ is precisely two to one if for each $k \in K$, there are exactly two points of X that map to k. For the remainder of the paper we are assuming that f is a precisely two to one mapping from N^* onto some (compact) space K. The question of whether there are non-trivial two to one maps on N^* is motivated by the papers of van Douwen [vD93] and R. Levy [Lev04]. In particular, Levy asks if every two to one image of N^* is homeomorphic to N^*. In fact there are several questions in [Lev04] that are consistently answered by the results in this paper. The behavior of two to one maps on N^* when CH is assumed is investigated in [DT04]. It is well known that van Douwen has shown that there is a compact separable space which is a ≤ 2 to one image of N^* and this pathology motivates the current study. R. Levy showed that if f is precisely two to one on N^* then K will have weight equal to $\mathfrak{c}$ and countable discrete subsets of K will have closure homeomorphic to βN.

Proposition 1. *If f is locally one to one (every point has a neighborhood on which f is one to one), then N can be partitioned into $a \cup b$ such that $f[a^*] = f[b^*] = K$. Since f is two to one, f is then a homeomorphism on each of a^* and b^*.*

Proof. If each point of N^* has a neighborhood on which f is one to one, then there is a finite cover by such neighborhoods. Let $\mathcal{A}$ be a finite partition of N such that f is one to one on each $a^* \in \mathcal{A}$. Enumerate $\mathcal{A} = \{a_i : i \leq n\}$. We will use induction on n. Consider the compact set $B_0 = f^{-1}(f[a_0^*]) \setminus a_0^*$ and note that $f[B_0] = f[a_0^*]$. Since f is two to one, $f[B_0 \cap a_1^*]$ is disjoint from $f[\bigcup_{1<j} a_j^*]$. Therefore there is a $c_1 \subset a_1$ such that $B_0 \cap a_1^* \subset c_1^*$ and $f[c_1^*]$ also disjoint from $f[\bigcup_{1<j} a_j^*]$. Since f is precisely two to one, and is one to one on c_1^*, it follows that $f[c_1^*] \subset f[a_0^*]$. That is, we have shown that $B_0 \cap a_1^* = c_1^*$. The same argument applies for each $i > 0$ replacing 1, hence B_0 is equal to b_0^* for some infinite $b_0 \subset N \setminus a_0$. It follows that the restriction of f to the union of $\{(a_1 \setminus b_0)^*, (a_2 \setminus b_0)^*, \ldots, (a_n \setminus b_0)^*\}$ is precisely two to one and is one to one on each piece. □

We will also assume throughout that PFA, OCA and MA hold. The statements of each of these can be found in [Tod89] and some familiarity will be assumed. Basic information about N^* can be found in [Wal74]. Of course it is well known

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(see [Vel93]) that OCA and MA implies that the mapping, $f^{-1} \circ f$ from a^* to b^* in the above lemma will actually be a trivial mapping.

Definition 2. Let $\mathcal{J}$ be the collection of those sets $a \in [\omega]^\omega$ on which f is precisely two to one and locally one to one.

Proposition 3. *For each $a \in \mathcal{J}$, there will be a permutation h_a on a such that h_a is never the identity, h_a^2 is the identity, and for each $b \subset a$, $f[b^*] = f[h(b)^*]$.*

Proof. By Proposition 1, a can be partitioned as $a_0 \cup a_1$ such that $f[a_0^*] = f[a_1^*]$. It follows easily that $f^{-1} \circ f$ restricts to a homemorphism from a_0^* to a_1^*. By OCA and MA, this is a trivial homeomorphism and h_a is the witness together with its inverse (with possibly finitely many elements of a removed). □

Naturally the task is to prove that $\mathcal{J}$ does not generate a proper ideal. The first step is to prove that $\mathcal{J}$ is not empty. We will proceed by first showing that if f is not locally one to one, then there is a point x such that for every countable family $\{A_n : n \in \omega\} \subset x$, there is an $A \in \mathcal{J}$ such that A is almost contained in each A_n.

There are two main results. The first is Lemma 14 which is critical to establishing that such an x exists. The second, Theorem 17, is to show that this leads to a contradiction.

We will certainly need the following results from [Far00, 3.3.2;3.8.2]

Definition 4. An ideal $\mathcal{I} \subset \wp(N)$ is ccc over fin, if there is no uncountable family of almost disjoint subsets of N such that none are in $\mathcal{I}$.

The following are consequences of OCA and MA (and therefore of PFA).

Proposition 5. *If $\Phi : \wp(N)/fin \to \wp(N)/fin$ is a homomorphism, then there is an $A \subset N$ and an $h : A \to N$, such that $\{a \subset N : \Phi(a) = h^{-1}(a)\}$ is ccc over fin.*

Proposition 6. [Far00, 3.5.5] *If a subset of N^* is a continuous image of N^*, then it is equal to the disjoint union of a clopen set and a nowhere dense set.*

It will be useful to explicit record the following corollary combining the previous two results.

Corollary 7. *If a nowhere dense set T of N^* is homeomorphic to N^*, then there does not exist an uncountable family of pairwise disjoint clopen subsets of N^* each of which meets T.*

2. Basic properties of f and K

We will have to show that K is nowhere ccc (which will just use MA). We fix some closed subset Z_0 of N^* such that f restricted to Z_0 is irreducible (meaning no proper closed subset of Z_0 will map onto). We will show that the image of $N^* \setminus Z_0$ is dense in K (i.e. f restricted to Z_0 is nowhere precisely two to one).

Lemma 8 (MA)**.** *For each clopen subset U of Z_0, there is a point $x \notin Z_0$ such that $f(x) \in f[U]$.*

Proof. Let U be a clopen subset of Z_0. If $f^{-1}(f[U])$ is contained in Z_0 then there is a $b_0 \in [N]^\omega$ such that $f^{-1}(f[b_0^*] \subset U$. Since f is irreducible, for each clopen $a \subset Z_0$, $J_a = a \cap f^{-1}(f[Z_0 \setminus a])$ is a nowhere dense subset of a. Let $\{b_\alpha : \alpha \in \mathfrak{c}\}$ enumerate $[b_0]^\omega$. For each $\alpha \in \mathfrak{c}\}$, $J_\alpha = J_{b_\alpha^*}$ is a nowhere dense subset of b_0^*. It is well known that MA implies that a clopen subset of N^* can not be covered by any family of $\mathfrak{c}$ many nowhere dense sets. Therefore there is a point $x \in b_0^*$ such that $x \notin \bigcup\{J_\alpha : \alpha \in \mathfrak{c}\}$. Since f is precisely two to one, there is a point $x' \neq x$ such that $f(x') = f(x)$. Since $x' \neq x$, there is an $\alpha \in \mathfrak{c}$ such that $x \in b_\alpha^*$ and $x' \notin b_\alpha^*$. Since $x \notin J_\alpha$, it follows that $x' \notin Z_0 \setminus b_\alpha^*$. Therefore $x' \notin Z_0$. □

Lemma 9. *For each $a \in [\omega]^\omega$, there is a $b \in [a]^\omega$ such that $f \restriction b^*$ is one-to-one. If f is one to one on b^* and $f[b^*]$ has interior, then $f^{-1}(f(b^*))$ contains c^* for some $c \in \mathcal{J}$.*

Proof. If a^* is not contained in Z_0, then any $b \subset a$ such that $b^* \cap Z_0$ will work. Otherwise we have that $a^* \subset Z_0$. Since f is irreducible on Z_0, $K \setminus f[Z_0 \setminus a*]$ is not empty. By Lemma 8, there is an $x \in N^* \setminus Z_0$ such that $f(x) \in K \setminus f[Z_0 \setminus a*]$. In fact, there is a element $C \in x$ such that $C \cap a = \emptyset$ and $f[C^*]$ is contained in $f[a^*]$. Since f is two to one, $f(x)$ is not in $f[N^* \setminus (a \cup C)^*]$. Let b be any subset of a such that $f[b^*] \cap f[C^*]$ and $f[b^*] \cap f[N^* \setminus (a \cup C)^*]$ are empty. It follows that b^* is a clopen subset of Z_0 with the property that $f[b^*]$ is disjoint from $f[N^* \setminus Z_0]$. This contradicts Lemma 8

Now assume that $f[b^*]$ has non empty interior in K. If b^* is disjoint from Z_0, fix any point $z_0 \in Z_0$ such that $f(z_0)$ is in the interior of $f[b^*]$. Find $b_0 \in z_0$ such that $b_0 \cap b$ is empty and $f[b_0^*]$ is a subset of $f[b^*]$. Clearly f is one to one on b_0^* and $f^{-1}(f(b^*))$ contains $f^{-1}(f(b_0^*))$, hence it suffices to show that the lemma holds if we assume that $b^* \cap Z_0$ is not empty.

Of course $b^* \cap Z_0$ is a clopen subset of Z_0. Since f is irreducible on Z_0, there is some $c_0 \subset b$ such that $c_0^* \cap Z_0$ is not empty and is such that $f[c_0^* \cap Z_0]$ is disjoint from $f[Z_0 \setminus b^*]$. Since f is one to one on b^*, it also follows that $f[c_0^*]$ is disjoint from $f[b^* \setminus c_0^*]$. Therefore $f[c_0^* \cap Z_0] = K \setminus (f[Z_0 \setminus b^*] \cup f[b^* \setminus c_0^*])$ is a clopen subset of K. Let $c \subset N$ be chosen so that $c^* = f^{-1}(f[c_0^* \cap Z])$. Since f is one to one on c_0^*, it follows that $c_0^* \cap c^* = c^* \cap Z_0$. This means that $c \setminus c_0$ and $c \cap c_0$ is the partition of c showing that $c \in \mathcal{J}$. □

For $b \subset N$, let f_{+b} denote the mapping $f \restriction b^*$ and $f_{-b} = f \restriction (N \setminus b)^*$.

Lemma 10. *If f is one to one on b^*, then $g_b = f_{-b}^{-1} \circ f_{+b}$ is an embedding of b^* into $(N \setminus b)^*$. If $c^* \subset g_b[b^*]$, then $c \cup b$ contains a member of $\mathcal{J}$.*

Proof. Since $g_b^{-1}[c^*]$ will be a clopen subset of b^*, there is a $c_0 \subset b$ such that $g_b[c_0^*] = c^*$. It follows that $f[c_0^*] = f[c^*]$ and that f is locally one to one on $(c \cup c_0)^*$. Therefore, $c \cup c_0$ will be in $\mathcal{J}$. □

Lemma 11 (MA)**.** *The space K is nowhere ccc.*

Proof. It suffices to show that Z_0 is nowhere ccc since the map is irreducible on Z_0. Let $A \subset N$ and $A^* \cap Z_0 \neq \emptyset$ and assume that $A^* \cap Z_0$ is ccc. Since $f[Z_0 \setminus A^*]$ and $f[A^* \cap Z_0]$ meet in a nowhere dense subset of K, there is a clopen subset b_0 of $A^* \cap Z_0$ such that $f[b_0] \cap f[Z_0 \setminus A^*]$ is empty. Since $Z_0 \setminus A^*$ is compact, there is a clopen set A_1^* of N^* such that $Z_0 \setminus A^* \subset A_1^*$ and $f[b_0] \cap f[A_1^*]$ is empty.

Let $\{b_\alpha : \alpha \in \mathfrak{c}\}$ enumerate the collection of clopen subsets of $Z_0 \cap A^*$. Further let $\{A_\alpha : \alpha \in \mathfrak{c}\}$ enumerate all the infinite subsets of N with the property that their closures are disjoint from Z_0.

For each clopen subset a of b_0, let $J_a = a \cap f^{-1}[f[Z_0 \setminus a]]$. Since f is irreducible on Z_0, each J_a is nowhere dense in Z_0. Also let $Y_\alpha = Z_0 \cap A^* \cap f^{-1}[f[A_\alpha^*]]$ for each

$\alpha \in \mathfrak{c}$. Since f is one to one on A^*_α, $f[A^*_\alpha]$ is nowhere ccc. Since we are assuming that $Z_0 \cap A^*$ is ccc, it follows that Y_α is also nowhere dense in Z_0.

We inductively define a family, $\{a_\beta : \beta < \alpha\}$ of clopen subsets of $A^* \cap Z_0$. Let $a_0 = b_0$. Our inductive hypotheses are that for each $\alpha < \mathfrak{c}$,

(1) $\{a_\beta : \beta < \alpha\}$ has the finite intersection property;
(2) a_β is contained in one of $\{b_\beta, b_0 \setminus b_\beta\}$;
(3) for each $\beta + 1 < \alpha$, $a_{\beta+1}$ is disjoint from $J_{a_\beta} \cup Y_\beta$; and
(4) the family $\{a_\beta : \beta < \alpha\}$ has non-empty intersection.

Suppose we have chosen the family $\{a_\beta : \beta < \alpha\}$. Let Z_α denote the closed set $\bigcap\{a_\beta : \beta < \alpha\}$. For each integer n, the selection of a_n is trivial, and since we are assuming that a_0 is ccc, we can assume that $\alpha \geq \omega$ and that Z_α is nowhere dense in Z_0. If α is a limit, then simply let a_α equal b_α if b_α meets Z_α, otherwise set $a_\alpha = b_0 \setminus b_\alpha$. Otherwise, $\alpha = \beta + 1$ and we must avoid $J_{a_\beta} \cup Y_\beta$. It suffices to show that Z_α is not contained in $J_{a_\beta} \cup Y_\beta$ because then we can select a_α to meet Z_α, miss $J_{a_\beta} \cup Y_\beta$ and to be contained in one of $\{b_\alpha, b_0 \setminus b_\alpha\}$.

Let $\mathcal{B}_\alpha$ be a Boolean subalgebra of $\{b_\gamma : \gamma < \mathfrak{c}\}$ which contains $\{a_\beta : \beta < \alpha\} \cup \{b_\alpha\}$. For each $\gamma < \alpha$, we can ensure that there is a countable subset $\mathcal{U}_\gamma$ of $\mathfrak{B}_\alpha$ such that each member of $\mathcal{U}_\gamma$ is disjoint from $J_\gamma \cup Y_\gamma$ and $\bigcup \mathcal{U}_\gamma$ is dense in $A^* \cap Z_0$. Since $\alpha < \mathfrak{c}$, there is such a $\mathcal{B}_\alpha$ of cardinality less than $\mathfrak{c}$. Since $\mathcal{B}_\alpha$ is ccc and of cardinality less than $\mathfrak{c}$, its Stone space is separable. Furthermore, $\mathcal{U}_\alpha$ generates a dense open subset, U_α, of the Stone space. Let D be a countable dense subset of U_α. The filter base on $\mathcal{B}_\alpha$ consisting of finite intersections of members of $\{a_\gamma : \gamma < \alpha\}$ generates a closed set, F_α, in $S(\mathcal{B}_\alpha)$ with character less than $\mathfrak{c}$. Martin's axiom implies that there is a countable set $\{d_n : n \in \omega\} \subset D$ which converges to F_α. That is, for any finite intersection of members of $\{a_\gamma : \gamma < \alpha\}$, there are only finitely many x_n which are not in the intersection. For each n, x_n is in fact a filter on $\mathcal{B}_\alpha$, hence there is a point $z_n \in Z_0$ which is also in each member of the filter.

For each n, let $z'_n \in N^*$ be distinct from z_n such that $f(z'_n) = f(z_n)$. Let $T = f[Z_\alpha]$, a nowhere dense subset of K. By construction, the image of $\overline{\{z_n : n \in \omega\}}$ is contained in $\{f(z_n) : n \in \omega\} \cup T$. Since K has no isolated points, and $\{f(z_n) : n \in \omega\}$ is a relatively closed subset of $Z_0 \setminus T$, it is nowhere dense and discrete. Therefore $f^{-1}(f(\overline{\{z_n : n \in \omega\}}))$ meets Z_0 in a nowhere dense set. Furthermore, this set contains $\overline{\{z'_n : n \in \omega\}}$.

Let $x \in Z_0$ be a limit point of $\{z_n : n \in \omega\}$, hence $x \in Z_\alpha$. There is a point x' which is a limit point of $\{z'_n : n \in \omega\}$ such that $f(x) = f(x')$ since $f(\overline{\{z_n : n \in \omega\}}) = f(\overline{\{z'_n : n \in \omega\}})$. Clearly $x \in b_\beta$ and we claim that $x \notin J_{a_\beta} \cup Y_\beta$. To show this it is sufficient (and necessary) to show that x' is not in $(Z_0 \setminus b_\beta) \cup A^*_\beta$. For each n, $z_n \notin Y_\beta$, hence $z'_n \notin A^*_\beta$. Since A^*_β is clopen and x' is a limit of $\{z'_n : n \in \omega\}$ it follows that $x' \notin A^*_\beta$. The collection $\{z'_n : n \in \omega\} \cap Z_0$ is contained in b_α, which is clopen in Z_0, hence this closure is disjoint from $Z_0 \setminus b_\alpha$. We will be finished if we show that the closure of $\{z'_n : n \in I\} = \{z'_n : n \in \omega\} \setminus Z_0$ is disjoint from Z_0. Since $\{z_n : n \in \omega\} \subset b_0$ and $f[b_0]$ is disjoint from $f[A^*_1]$, it follows that $\overline{\{z'_n : n \in \omega\}}$ is disjoint from A^*_1. Recall that we showed above that $f^{-1}(f(\overline{\{z_n : n \in \omega\}}))$ meets Z_0 in a nowhere dense set and that it contains $\overline{\{z'_n : n \in \omega\}} \cap Z_0$. Since it is nowhere dense and b_0 is ccc, there is a collection $\{c_n : n \in \omega\}$ of clopen subsets of N^* such that each is disjoint from $\{z'_n : n \in \omega\}$ and such that b_0 is contained in the closure of $\bigcup\{c_n \cap b_0 : n \in \omega\}$. For each $n \in I$, let d_n be a clopen subset of $N^* \setminus A_1^*$ such that $z'_n \in d_n$ and $d_n \cap (Z_0 \cup \bigcup\{c_k : k \leq n\})$ is empty. For each $n \in \omega$, shrink c_n by removing $\bigcup\{d_k : k < n\}$; note that this does not change $c_n \cap Z_0$. Then we have that $\bigcup_{n\in I} d_n$ and $A_1^* \cup \bigcup_n c_n$ are disjoint, and as is well-known, they have disjoint closures in N^*. Since the latter closure conains Z_0 we have finished the proof. □

Proposition 12. *If f is one to one on b^*, then b can be partitioned into two, b_0 and b_1, such that $f[b_0^*]$ is clopen, and $f[b_1^*]$ is nowhere dense.*

Proof. Again let g_b denote the embedding of b^* into $(N \setminus b)^*$ given by $f_{-b}^{-1} \circ f_{+b}$. By Proposition 6, there is $c \subset N \setminus b$ such that $c^* \subset g_b[b^*]$ and $g_b[b^*] \setminus c^*$ is nowhere dense. There are $b_0 \subset b$ such that $g_b[b_0^*] = c^*$ and $b_1 = b \setminus b_0^*$ will satisfy $g_b[b_1^*]$ is nowhere dense. It follows that $f[b_0^*] = f[c^*]$ and $f[b_0^*] = K \setminus f[(N \setminus (b_0 \cup c)^*]$ is clopen. In addition, $f[b_1^*]$ is nowhere dense in K because $g_b[b_1^*]$ is equal to $f_{-b}^{-1}(f[b_1^*])$ and is nowhere dense in $(N \setminus b)^*$. □

3. TREE-LIKE FAMILIES

An embedding of N^* into N^* is said to be trivial, if the embedding lifts to an embedding of βN into βN. It is an open problem to determine if there can be a non-trivial embedding of N^* into N^* under PFA. If there are none, then it is easy to use Levy's result in [Lev04] to show that the set b_1 in Proposition 12 would be empty. The main result of this section is used as an alternative approach.

Proposition 13. [Vel93, 2.3] *Let $\mathcal{A}$ be an uncountable almost disjoint family of infinite subsets of N. Then there is an uncountable $\mathcal{B} \subset \mathcal{A}$ and for each $a \in \mathcal{B}$ a partitition $a = a_0 \cup a_1$ such that the family $\mathcal{B}_i = \{a_i : a \in \mathcal{B}\}$ is tree-like for each $i \in \{0, 1\}$.*

Lemma 14. *Suppose that $\{a_\alpha : \alpha \in \omega_1\}$ is a tree-like family of subsets of ω with the property there is a $b_\alpha \in [a_\alpha]^\omega$ such that $f(b_\alpha^*)$ is disjoint from $f[(N \setminus a_\alpha)^*]$. Then there is an $b_\alpha = b$ such that $g_b[b^*]$ has interior.*

Proof. We may assume that f is one-to-one on b_α^* by Lemma 9. For each $c \subset a_\alpha$ define $F(c) \subset b_\alpha$ as follows. Since $f[(a_\alpha \setminus b_\alpha)^*)]$ contains $f(b_\alpha^*)$ and f is precisely two to one, there will be a subset $F(c)$ of b_α such that

$$f[F(c)^*] = f(b_\alpha^*) \cap f[(c \setminus b_\alpha)^*] \ .$$

The definition of F on $\wp(a_\alpha)$ can also be expressed as $F(c)^*$ is the clopen subset of b^* which is equal to $g_b^{-1}[c^* \cap g_b(b^*)]$.

It is easily seen that F is a homomorphism from $\wp(a_\alpha)/fin$ onto $\wp(b_\alpha)/fin$. By Corollary 7, we need to find some α and some uncountable family of pairwise disjoint clopen sets each of which meets $g_{b_\alpha}[b_\alpha^*]$. Equivalently, by Farah [Far00] (Proposition 5), if the kernel of F is not ccc over fin, then there is a $c \subset a_\alpha \setminus b_\alpha$ such that F is a trivial isomorphism from $\wp(c)$ to $\wp(b_\alpha)$. We proceed as in [Vel93].

Let $\mathcal{X}$ denote the set of all pairs $\langle c, d\rangle$ such that for some α, $d \subset c \subset a_\alpha \setminus b_\alpha$, and each of $F(d)$ and $F(c \setminus d)$ are not 0.

We define a set $K_0 \subset [\mathcal{X}]^2$ according to $(\langle c, d\rangle, \langle \bar{c}, \bar{d}\rangle) \in K_0$ providing

(1) $c \subset a_\alpha$ and $\bar{c} \subset a_{\bar{\alpha}}$ implies $\alpha \neq \bar{\alpha}$
(2) $c \cap F(\bar{c})$ and $\bar{c} \cap F(c)$ are empty;
(3) $c \cap \bar{d} = \bar{c} \cap d$;
(4) $F(c) \cap F(\bar{d})$ is not equal to $F(\bar{c}) \cap F(d)$.

The appropriate separable metric topology on $\mathcal{X}$ (given by considering it as embedded in $\wp(\omega)^4$ by the mapping sending $\langle c,d\rangle$ to $\langle c,d,F(c),F(d)\rangle$) will result in K_0 being an open subset of $[\mathcal{X}]^2$ (see [Vel93]).

Assume that $\mathcal{Y}$ is an uncountable subset of $\mathcal{X}$ and that $[\mathcal{Y}]^2\subset K_0$. Let $I\subset\omega_1$ be the set of α such that there is $\langle c,d\rangle\in\mathcal{Y}$ such that $c\subset a_\alpha$. Also let $\langle c_\alpha,d_\alpha\rangle\in\mathcal{Y}$ be chosen for each $\alpha\in I$ so that $c_\alpha\subset a_\alpha$. Since $[\mathcal{Y}]^2\subset K_0$ and $\mathcal{Y}$ is uncountable, it follows that I is uncountable.

Let $C=\bigcup\{c_\alpha:\alpha\in I\}$ and $D=\bigcup\{d_\alpha:\alpha\in I\}$. Let $\langle c,d\rangle$ and $\langle\bar c,\bar d\rangle$ be an arbitrary distinct pair from $\mathcal{Y}$. Note that $c\cap(F(c)\cup F(\bar c))$ is empty. It follows that C is disjoint from $\bigcup\{F(c):(\exists d)\langle c,d\rangle\in\mathcal{Y}\}$. Also, $D\cap c$ will equal d for each $\langle c,d\rangle\in\mathcal{Y}$. Hence $(C\setminus D)\cap c=c\setminus d$ for each $\langle c,d\rangle\in\mathcal{Y}$.

Now consider the two families $\{F(d_\alpha):\alpha\in I\}$ and $\{F(c_\alpha\setminus d_\alpha):\alpha\in I\}$. Assume that $E\subset\omega$ and that $E\cap F(c_\alpha)=^* F(d_\alpha)$ for each $\alpha\in I$. Let $n\in\omega$ and $I'\in[I]^{\omega_1}$ such that $(E\cap F(c_\alpha))\Delta F(d_\alpha)$ is contained in n for all $\alpha\in I'$. Let $\alpha\neq\beta$ both be in I'. We may assume that $F(c_\alpha)\cap n=F(c_\beta)\cap n$, $F(d_\alpha)\cap n=F(d_\beta)\cap n$. Also, we may assume that $F(d_\alpha)\setminus F(c_\alpha)$ is n contained in n for all $\alpha\in I'$.

Suppose that $j\in F(c_\alpha)\cap F(d_\beta)$ and $j\notin F(c_\beta)\cap F(d_\alpha)$. Clearly j must be larger than n. Since $j\in F(d_\beta)$, it follows that $j\in F(c_\beta)$. Therefore j is in E. On the other hand, since j is in $E\cap F(c_\alpha)$, it must follow that $j\in F(d_\alpha)$, contradicting that $j\notin F(c_\beta)\cap F(d_\alpha)$. .

It follows then that there is no such E. This means that $\bigcup\{F(d_\alpha)^*:\alpha\in I\}$ and $\bigcup\{F(c_\alpha\setminus d_\alpha)^*:\alpha\in I\}$ do not have disjoint closures in N^*. Fix any $x\in N^*$ which is in each of the closures. Notice that $x\notin C^*$.

Since $f[d_\alpha^*]$ is equal to $f[F(d_\alpha)^*]$ and $f[(c_\alpha\setminus d_\alpha)^*]=f[(F(c_\alpha\setminus d_\alpha))^*]$, it follows that $f(x)$ is in the image of the closure of $\bigcup_{\alpha\in I}d_\alpha^*$ and of $\bigcup_{\alpha\in I}(c_\alpha\setminus d_\alpha)^*$. Therefore $f(x)$ is in the image of D^* and of $(C\setminus D)^*$. However this contradicts that $f(x)$ only has two points mapping to it, we have found points in $D^*,(C\setminus D)^*$, and $(N\setminus C)^*$.

Therefore by OCA, $\mathcal{X}$ can be expressed as a countable union $\bigcup_n\mathcal{Y}_n$ such that $[\mathcal{Y}_n]^2\cap K_0$ is empty for each n. For each n, there is a countable $Y_n\subset\mathcal{Y}_n$ such that for each integer m and each $\langle c,d\rangle\in\mathcal{Y}_n$, there is some $\langle\bar c,\bar d\rangle\in Y_n$ such that $c\cap m=\bar c\cap m$, $d\cap m=\bar d\cap m$, $F(c)\cap m=F(\bar c)\cap m$, and $F(d)\cap m=F(\bar d)\cap m$.

Fix any $\alpha\in\omega_1$ such that $\bar c\cap a_\alpha$ is finite for each $\langle\bar c,\bar d\rangle\in\bigcup_n Y_n$. Construct an increasing sequence $\{k_n:n\in\omega\}$ of integers so that for each n and each $i\leq n$ and each sequence c',d',a',b' of subsets of k_n, if there is a $\langle c,d\rangle\in\mathcal{Y}_i$ such that $c\cap k_n=c'$, $d\cap k_n=d'$, $F(c)\cap k_n=a'$, and $F(d)\cap k_n=b'$, then there is a pair $\langle c,d\rangle\in Y_i$ that also has this property, and in addition, $a_\alpha\cap a_{\bar\alpha}\subset k_{n+1}$ where $c\subset a_{\bar\alpha}$.

Define E_i to be $\bigcup\{a_\alpha\cap[k_{3n+i},k_{3n+i+1}):n\in\omega\}$ for $i\in 3$. There is an $i\in 3$ such that $F(E_i)$ is not finite. There is a $j\in 3$ such that $E_j\cap F(E_i)$ is not finite. Fix any $c\subset E_i$ such that $F(c)$ is infinite and is contained in E_j. Let d_0,d_1,d_2 be a partition of c so that $F(d_0)$, $F(d_1)$, and $F(d_2)$ are each infinite. For simplicity we will assume that $\{i,j\}=\{1,2\}$. Note that for all $x\subset d_1$, the pair $\langle c,d_0\cup x\rangle$ is a member of $\mathcal{X}$ (since $c\setminus(d_0\cup x)$ contains d_2).

Under the same (or similar) circumstances it is shown in [Vel93, 2.2] that $F\restriction\wp(a_\alpha)$ is Borel. Then one can apply the results in [Far00] to deduce that $g_{b_\alpha}[b_\alpha^*]$ is not nowhere dense. However, rather than checking that the approach in [Vel93] is applicable, we will directly produce an uncountable family of clopen subsets of N^* each of which will meet $g_{b_\alpha}[b_\alpha^*]$.

We construct an increasing sequence $\{m_i : i \in \omega\} \subset \{k_{3\ell} : \ell \in \omega\}$ together with subsets $t_i \subset d_1 \cap [m_i, m_{i+1})$ and possibly infinite sets $\{J_i : i \in \omega\}$ by induction. We can let $m_0 = 0$ and $J_{-1} = \emptyset$. For each $J \subset N$, let $D(J) = d_0 \cup (d_1 \cap \bigcup\{[k_{3j}, k_{3j+3}) : j \in J\})$.

Given that m_i and J_{i-1} have been chosen we will construct the set t_i also by a finite induction and will then choose m_{i+1}. Fix an enumeration $\{(e_\ell, n_\ell, n'_\ell) : \ell < L\}$ of $\wp(m_i) \times i \times i$. We will construct $\{t(i,\ell) : \ell < L\}$ such that $t(i+1,\ell) \cap \max(t(i,\ell)) + 1 = t(i,\ell)$ and will let $t_i = \bigcup\{t(i,\ell) : \ell < L\}$. Our inductive hypotheses on J_i are that $F(D(N \setminus J_i)) \setminus F(d_0)$ is infinite and that if $j \in J_i \setminus J_{i-1}$, then $k_{3j} > m_i$.

As we define $t(i,\ell)$, we will also define $J(i,\ell)$ and will set $J_i = \bigcup\{J(i,\ell) : \ell < L\}$. For convenience let $t(i,-1) = \emptyset$ and $J(i,-1) = J_{i-1}$.

Suppose we have chosen $t(i,\ell-1)$ and $J(i,\ell-1)$ such that $F(D(N \setminus J(i,\ell-1))) \setminus F(d_0)$ is infinite.

First choose, if possible, $J'(i,\ell) \supset J(i,\ell-1)$ such that $J'(i,\ell) \cap \max(t(i,\ell-1)) = J(i,\ell-1)$, $\langle c, D(J'(i,\ell))\rangle \in \mathcal{Y}_{n'}$, and $F(D(N \setminus J'(i,\ell))) \setminus F(d_0)$ is infinite. If there is no such $J'(i,\ell)$, then let $J'(i,\ell) = J(i,\ell-1)$.

Next choose, if possible, an $x \subset d_1 \setminus \max(t(i,\ell-1))$ such that

$$F(e_\ell \cup t(i,\ell-1) \cup D(J'(i,\ell)) \cup x) \setminus F(D(J'(i,\ell-1)))$$

is infinite and $\langle c, d_0 \cup e_\ell \cup t(i,\ell-1) \cup D(J(i,\ell-1)) \cup x\rangle \in \mathcal{Y}_{n_\ell}$.

If such an x exists, call this Case one, and choose some large enough $j' \in x$ such that there is a $k_{3\ell'} < j'$ such that $\ell' \notin J'(i,\ell)$ and

$$F(e_\ell \cup t(i,\ell-1) \cup D(J'(i,\ell)) \cup x) \setminus F(D(J'(i,\ell)))$$

contains some element of $F(c) \cap [m_i, k_{3\ell'})$. We define $J(i,\ell) = J'(i,\ell)$, and $t(i,\ell) = t(i,\ell-1) \cup (x \cap j+1)$. Note that ℓ' will not be in J_i.

On the other hand, if no such an x exists, then choose $J(\ell,i) \supset J'(\ell,i)$ such that each of

$$F(e_\ell \cup t(i,\ell-1) \cup D(J(i,\ell))) \setminus F(D(J'(i,\ell)))$$

and

$$F(D(N \setminus J(i,\ell)))$$

are infinite. Again choose an $\ell' \notin J'(i,\ell)$ so that $k_{3\ell'} > \max(t(i,\ell-1))$, and ensure that $J(i,\ell) \cap \ell'+1$ equals $J'(i,\ell) \cap \ell'$. Also ensure that $d_1 \cap [k_{3\ell'}, k_{3\ell'+3})$ is not empty and let $t(i,\ell) = t(i,\ell-1) \cup (d_1 \cap [k_{3\ell'}, k_{3\ell'+3}))$.

Let $m_{i+1} = k_{3n}$ be chosen so that $3n \notin J_i = \bigcup\{J(i,\ell) : \ell < L\}$ and $t_i = \bigcup\{t(i,\ell) : \ell < L\} \subset m_{i+1}$.

Let $J_\omega = \bigcup_i J_i$ and note that, by construction $D(J_\omega) \cap m_{i+1} = D(J_i) \cap m_{i+1}$. For each infinite $I \subset \omega$, set $e_I = \bigcup\{t_i : i \in I\} \subset d_1$. We finish by showing that the clopen set e_I^* meets $g_{b_\alpha}[b_\alpha^*]$. To show this, it suffices to show that $F(e_I)$ is infinite. Since $F(e_I)$ contains $F(D(J_\omega) \cup e_I) \setminus F(D(J_\omega))$ mod finite, it suffices to show this latter set is infinite.

Since $\langle c, D(J_\omega) \cup e_I\rangle \in \mathcal{X}$, there is an n such that $\langle c, D(J_\omega) \cup e_I\rangle \in \mathcal{Y}_n$. There is also an n' such that $\langle c, D(J_\omega)\rangle \in \mathcal{Y}_{n'}$. Let $i \in I$ be any integer greater than both n and n'. We will show that $F(D(J_\omega) \cup e_I) \setminus F(D(J_\omega))$ is not contained in i.

Set $d = D(J_\omega) \cup e_I$ and fix ℓ such that at stage i in the construction of the m_i's, $e_\ell = d \cap m_i$ and $n_\ell = n$, and $n'_\ell = n'$. We consider the properties of $t(i,\ell)$. Let $Y = D(J'(i,\ell))$. Since $\langle c, D(J_\omega)\rangle$ is in $\mathcal{Y}_{n'}$, it follows that we were able to ensure that $\langle c, D(J'(i,\ell))\rangle = \langle c, Y\rangle$ is in $\mathcal{Y}_{n'}$.

Recall that there was an ℓ' such that $m_i < k_{3\ell'} < m_{i+1}$ and that the maximum of $t(i,\ell)$ was greater than $k_{3\ell'}$. If $J(i,\ell) \setminus J'(i,\ell)$ was infinite, then we know $F(e_\ell \cup t(i,\ell-1) \cup D(J(i,\ell))) \setminus F(Y)$ is infinite. Therefore $F(d) \setminus F(Y)$ is infinite because $d \supset e_\ell \cup t(i,\ell-1) \cup D(J(i,\ell))$. If we set $\bar{x} = d \setminus \max(t(i,\ell-1))$, then $\bar{x}$ would be a witness to the fact that we should have been in case one when defining $t(i,\ell)$. Therefore $J(i,\ell) = J'(i,\ell)$ and at stage ℓ we were able to find some x as in Case one. In addition, $t(i,\ell)$ was defined as $t(i,\ell-1) \cup (x \cap j+1)$ for some $j \in x \setminus k_{3\ell'}$.

Since $e_\ell = d \cap m_i$ and $i \in I$, we have that

$$d \cap k_{3\ell'} = (e_\ell \cup t(i,\ell-1) \cup D(J'(i,\ell)) \cup x) \cap k_{3\ell'}.$$

That is, if we let $d_x = e_\ell \cup t(i,\ell-1) \cup D(J'(i,\ell)) \cup x$, then $\langle c, d_x\rangle \in \mathcal{Y}_n$ and $d_x \cap k_{3\ell'} = d \cap k_{3\ell'}$.

Now since $\langle c,d\rangle \in \mathcal{Y}_n$, there is a pair $\langle \bar{c}, \bar{d}\rangle \in Y_n$ such that $\bar{c} \cap k_{3\ell'} = c \cap k_{3\ell'}$, $\bar{d} \cap k_{3\ell'} = d \cap k_{3\ell'}$, $F(\bar{c}) \cap k_{3\ell'} = F(c) \cap k_{3\ell'}$, and $F(\bar{d}) \cap k_{3\ell'} = F(d) \cap k_{3\ell'}$. In addition, if we let $\bar{\alpha} \in \omega_1$ such that $\bar{c} \subset a_{\bar{\alpha}}$, we have that $a_{\bar{\alpha}} \cap a_\alpha \subset k_{3\ell'+1}$. Further, recall that $(c \cup F(c)) \cap [k_{3\ell'}, k_{3\ell'+1})$ is empty. Observe also that $Y \cap k_{3\ell'}$ is equal to $D(J_\omega) \cap k_{3\ell'}$.

It follows then that each of

(1) $c \subset a_\alpha$ and $\bar{c} \subset a_{\bar{\alpha}}$ and $\alpha \neq \bar{\alpha}$
(2) $c \cap F(\bar{c})$ and $\bar{c} \cap F(c)$ are empty: because $c \cap F(\bar{c}) \subset c \cap (F(\bar{c}) \cap a_\alpha) \subset c \cap (F(\bar{c}) \cap k_{3\ell'+1}) \subset c \cap F(c)$ and similarly, $\bar{c} \cap F(c) \subset \bar{c} \cap F(c) \cap k_{3\ell} \subset c \cap F(c)$
(3) $c \cap \bar{d} = \bar{c} \cap d$: because each of $c \cap \bar{d}$ and $\bar{c} \cap d$ are equal to $d \cap k_{3p+\ell}$

Therefore, the only reason that $(\langle c,d\rangle, \langle \bar{c}, \bar{d}\rangle)$ is not in K_0 is that $F(c) \cap F(\bar{d})$ must be equal to $F(\bar{c}) \cap F(d)$. The same is true with d_x in place of d, hence $F(c) \cap F(\bar{d})$ must be equal to $F(\bar{c}) \cap F(d_x)$. By the choice of x, there is some integer $m' \in F(c) \cap [m_i, k_{3\ell'}) \cap F(d_x) \setminus F(Y)$, and therefore $m' \in F(\bar{c}) \cap F(\bar{d})$. This then means that $m' \in F(d)$.

By the same reasoning, there is a pair $\langle c', d'\rangle \in Y_{n'}$ such that $c' \cap k_{3\ell'} = c \cap k_{3\ell'}$, $d' \cap k_{3\ell'} = D(J_\omega) \cap k_{3\ell'}$, $F(c') \cap k_{3\ell'} = F(c) \cap k_{3\ell'}$, and $F(d') \cap k_{3\ell'} = F(D(J_\omega)) \cap k_{3\ell'}$. In addition, if we let $\alpha' \in \omega_1$ such that $c' \subset a_{\alpha'}$, we have that $a_{\alpha'} \cap a_\alpha \subset k_{3\ell'+1}$. Further, recall that $(c \cup F(c)) \cap [k_{3\ell'}, k_{3\ell'+1})$ is empty. Repeating the argument above with $\langle c, D(J_\omega)\rangle$ and $\langle c', d'\rangle$ in place of $\langle c,d\rangle$ and $\langle \bar{c}, \bar{d}\rangle$ respectively, we have that $F(D(J_\omega)) \cap k_{3\ell'}$ is equal to $F(d') \cap k_{3\ell'}$. Furthermore, $F(Y) \cap k_{3\ell'}$ will also equal $F(d') \cap k_{3\ell'}$ because $Y \cap k_{3\ell'} = D(J_\omega) \cap k_{3\ell'}$. Since $m' \in F(c) \cap F(c') \setminus F(Y)$, it follows that $m' \notin F(D(J_\omega))$.

This shows that $m' \in F(d) \setminus F(D(J_\omega))$ and completes the proof. □

Corollary 15. *The space Z_0 is regular closed in N^*.*

Proof. Let $A \subset N$ be any infinite set such that $A^* \cap Z_0$ is not empty. Since f is irreducible on Z_0, $f[A^* \cap Z_0]$ has interior in K. By Lemma 11, there is an uncountable family $\{U_\alpha : \alpha \in \omega_1\}$ of pairwise disjoint open subsets of $f[A^* \cap Z_0]$. For each α, let c_α be an infinite set such that $f[c_\alpha^*]$ is a subset of U_α and such that f is one to one on c_α^*. If any of the c_α's are such that $f[c_\alpha^*]$ has interior then we can apply Lemma 9.

On the other hand, we assume that each c_α^* is disjoint from Z_0. Fix an a_α such that $f[a_\alpha^*]$ is a subset of U_α and meets $f[c_\alpha^*]$. By Proposition 13, we may assume that $\{a_\alpha : \alpha \in \omega_1\}$ is tree-like. It follows that $f[N^* \setminus (a_\alpha \cup c_\alpha)^*]$ is a closed subset of K which does not contain $f(c_\alpha^*)$. Let b_α be any infinite subset of a_α such that

$f[b_\alpha^*]$ is disjoint from $f[N^* \setminus (a_\alpha \cup c_\alpha)^*]$ and the nowhere dense set $f[c_\alpha^*]$. Now we can apply Lemma 14 □

Lemma 16. *If f is not locally one to one, then there is a point x such that for every countable family $\{A_n : n \in \omega\} \subset x$, there is an $A \in \mathcal{J}$ such that A is almost contained in each A_n.*

Proof. We will choose a point x and an infinite set $b \subset N$, such that $b \notin x$, $f(x) \in f[b^*]$, and f is not one to one on any A^* with $A \in x$.

First assume that there is some infinite $b \subset N$ such that $f[b^*]$ is nowhere dense in K. Since $f[(N \setminus b)^*]$ is equal to K, we can fix $x \in N^*$, such that $f(x) \in f[b^*]$.

Otherwise, we may assume that $f[b^*]$ has interior for all infinite $b \subset N$. In this case, fix any $x \in N^*$ such that $f \restriction A^*$ is not one to one for any $A \in x$. Let $b \subset N$ be any infinite set such that $f(x) \in f[b^*]$ and $b \notin x$.

Assume that $\{A_n : n \in \omega\} \subset x$. We may assume that for each n, $A_n \cap b$ is empty and $A_{n+1} \subset A_n$. It follows that $f(x) \notin f[(N \setminus (b \cup A_n))^*]$ for each n. By continuity of f, there is a $B_n \in x$ such that $f[B_n^*]$ is disjoint from $f[(N \setminus (b \cup A_n))^*]$. Let A be almost contained in B_n for each n. It follows that $f[A^*] \cap f[(N \setminus (b \cup A_n))^*]$ is empty for each n. We may assume that f is one to one on A^* by Lemma 9. Furthermore, if $f[A^*]$ has interior in K, then by Lemma 9, there is a $c \in \mathcal{J}'$ such that $f[c^*] \subset f[A^*]$. Therefore $f[c^*]$ is disjoint from $f[(N \setminus (A_n \cup b))^*]$ for each n, hence c is almost contained in A_n for all n.

Now we consider the case that $f[A^*]$ is nowhere dense for each A almost contained in all the A_n's. Since Z_0 is regular closed, we can choose, for each n, an infinite $b_n \subset B_n$ such that $b_n^* \subset Z_0$. We may further suppose that there is a disjoint set c_n such that $f[b_n^*] = f[c_n^*]$. It follows that there is some $m > n$ such that $b_n^* \cap B_m^*$ is empty (otherwise there would be an $A \subset b_n$ almost contained in each A_n such that $f[A^*]$ had interior).

We may also assume that b_n is disjoint from b_m for each $n \neq m$. For each n, let $\{b(n,\alpha) : \alpha \in \omega_1\}$ be an almost disjoint family of infinite subsets of b_n. Inductively define an almost disjoint family $\{a_\alpha : \alpha \in \omega_1\}$ of subsets of $\bigcup_n b_n$ so that $a_\alpha \cap b_n = b(n,\alpha)$ for each n. Apply Proposition 13, to find a partition a_α as $a_\alpha^0 \cup a_\alpha^1$ so that there is an uncountable $I \subset \omega_1$ such that each of the families $\{a_\alpha^0 : \alpha \in I\}$ and $\{a_\alpha^1 : \alpha \in I\}$ are tree-like. For each $\alpha \in I$, at least one of a_α^0 or a_α^1 will meet infinitely many of the b_n's. Therefore we can assume we have an uncountable tree-like family $\{a_\alpha : \alpha \in \omega_1\}$ such that each a_α meets infinitely many of the b_n's in an infinite set.

Assume that $f[(N \setminus a_\alpha)^*]$ contains $f[a_\alpha^*]$ for some α. It then follows that f is one to one on a_α^*. By Proposition 12, we can write a_α as $a_\alpha^0 \cup a_\alpha^1$ so that $f[(a_\alpha^1)^*]$ is nowhere dense and $f[(a_\alpha^0)^*]$ is clopen. It follows then that the clopen set $f[(a_\alpha \cap b_n)^*]$ is contained in $f[(a_\alpha^0)^*]$ for each n. This however contradicts the current assumption that $f[A^*]$ is nowhere dense for any $A \subset a_\alpha$ which is almost disjoint from each b_n. From this we conclude that there is some infinite $b_\alpha \subset a_\alpha$ such that $f[b_\alpha^*]$ is disjoint from $f[(N \setminus a_\alpha)^*]$.

Finally, the theorem follows by applying Lemma 14. □

4. LOCALLY ONE TO ONE

Theorem 17 (PFA). *The function f is locally one to one.*

Proof. Assume that f is not locally one to one and choose an x as in Lemma 16. For each $a \in \mathcal{J}$, we have the permutation h_a on a such that h_a is never the identity and h_a^2 is the identity on a (3).

We define a poset P by $p \in P$ if there is a finite set $F_p \subset \omega_1$, a set $\mathcal{M}_p = \langle M_\alpha : \alpha \in F_p \rangle$ of elementary submodels, a finite function $\langle a_\alpha : \alpha \in F_p \rangle$ whose range is a subset of $\mathcal{J}'$, and a finite function s_p with domain n_p into $\{0, 1, 2\}$.

For each $\alpha \in F_p$, we require that M_α is countable elementary submodel of a suitable $H(\theta)$ (containing $x, \mathcal{J}', f$). We require that $a_\alpha \in \mathcal{J}$ is contained mod finite in each $A \in x \cap M_\alpha$. For $\beta > \alpha$ also in F_p, $M_\alpha \in M_\beta$. For convenience, we can require that $M_\alpha \cap \omega_1 = \alpha$. The final condition is that there must be an $i \in a_\alpha$ and a $j \in a_\beta$ such that $s_p(i) = 0$, $s_p(j) = 1$, $h_{a_\alpha}(i) = h_{a_\beta}(j)$, and $s_p(h_{a_\alpha}(i)) = 2$.

We show below that P is a proper (Lemma 20) and that for each $\alpha \in \omega_1$, the set $D_\alpha = \{p \in P : F_p \setminus \alpha \neq \emptyset\}$ is dense (Lemma 19). Suppose that $G \subset P$ is a filter which meets D_α for all $\alpha \in \omega_1$. Let $I = \bigcup\{F_p : p \in G\}$ and $\{a_\alpha : \alpha \in I\}$ be the corresponding elements of $\mathcal{J}$. For each $\alpha \in I$, let $h_\alpha = h_{a_\alpha}$. Let $A_0 = \{n \in \omega : (\exists p \in G) s_p(n) = 0\}$, $A_1 = \{n \in \omega : (\exists p \in G) s_p(n) = 1\}$ and $A_2 = \{n : (\exists p \in G) s_p(n) = 2\}$. For each $\alpha \in I$, let $b(\alpha, 0) = A_0 \cap a_\alpha$ and $b(\alpha, 1) = A_1 \cap a_\alpha$. Let $c(\alpha, 0) = A_2 \cap h_\alpha(b(\alpha, 0))$ and $c(\alpha, 1) = A_2 \cap h_\alpha(b(\alpha, 1))$.

Assume that $d \subset A_2$ is such that $c(\alpha, 0) \subset^* d$ and $d \cap c(\alpha, 1) =^* \emptyset$ for all $\alpha \in I$. There is some $m \in \omega$ and an uncountable $I' \subset I$ such that $c(\alpha, 0) \subset d \cup m$ and $d \cap c(\alpha, 1) \subset m$. We can also assume that there is an integer $m' \geq m$, a finite set $a' \subset m'$ and a function h from a' into ω such that $a_\alpha \cap m' = a'$, $h_\alpha^{-1}(m) \subset a'$, and $h_\alpha \restriction a' = h$ for all $\alpha \in I$. Let $\alpha < \beta \in I'$ and $p \in G$ such that $\{\alpha, \beta\} \subset F_p$. By the definition of P, there is an $i \in a_\alpha$, $j \in a_\beta$ such that $s_p(i) = 0$, $s_p(j) = 1$, $h_\alpha(i) = h_\beta(j)$, and $s_p(h_\alpha(i)) = 2$. Clearly $i \in b(\alpha, 0)$ and $j \in b(\beta, 1)$. We want to show that each of i, j and $k = h_\alpha(i)$ are larger than m. Since $h_\alpha^{-1}(k)$ is different than $h_\beta^{-1}(k)$ it follows that $k \geq m$. Since h_α and h_β disagree on i and j, they too are greater than m. Therefore, $k \in c(\alpha, 0) \setminus m$, hence $k \in d$. However this contradicts that $d \cap c(\beta, 1) \subset m$.

It follows that the sets $\bigcup\{c(\alpha, 0)^* : \alpha \in I\}$ and $\bigcup\{c(\alpha, 1)^* : \alpha \in I\}$ do not have disjoint closures in N^*. Let x be a common limit point. Since $f[\bigcup\{b(\alpha, 0)^* : \alpha \in I\}]$ contains $f[\bigcup\{c(\alpha, 0)^* : \alpha \in I\}]$, it follows that there is a point $y \in A_0^*$ such that $f(y) = x$. Similarly there is a point $z \in A_1^*$ such that $f(z) = x$. Since $x \in A_2^*$, we have produced three distinct points mapping to the same element of K. $\square$

For a set $\mathcal{A} \subset \wp(N)$, and a set $a \subset N$, let $a \prec^* \mathcal{A}$ indicate that $a \subset^* A$ for each $A \in \mathcal{A}$.

Lemma 18. *Assume that $\mathcal{S} \subset \mathcal{J}$ and that for each countable subset $\mathcal{A}$ of x, there is an $a \in \mathcal{S}$ such that $a \prec^* \mathcal{A}$ Then the set S consisting of all k satisfying*

$$(\forall \mathcal{A} \in [x]^\omega)(\exists a \in \mathcal{S}) \ (a \prec^* \mathcal{A} \text{ and } (k \in a)\}$$

is a member of x.

Proof. Let $x, \mathcal{S} \in M$ be any countable elementary submodel and let $\mathcal{S}'$ denote the elements of $\mathcal{S}$ which are mod finite contained in every member of $x \cap M$. It follows by elementarity that $B = \{k : (\exists a \in \mathcal{S}') k \in a\}$ is contained in S. If $B \notin x$, then there is some $a \in \mathcal{S}'$ such that $a \subset^* N \setminus B$. This is clearly impossible since, by definition, a is a subset of B. Therefore $S \in x$. $\square$

Lemma 19. *For each $\alpha \in \omega_1$, the set D_α is a dense subset of P.*

Proof. Let $p \in P$ be arbitrary and let $p, \alpha, \mathcal{J} \in M$ for some countable elementary submodel of $H(\lambda)$. Let $\delta = M \cap \omega_1$ and set $M_\delta = M$.

If F_p is empty, then we can simply apply Lemma 9, to find an $a_\delta \in \mathcal{J}$ such that $a_\delta \subset^* A$ for all $A \in x \cap M$. and set $p' = (\delta, \langle M \rangle, \langle a_\delta \rangle, s_p)$. It follows easily that $p' \leq p$ and $p' \in D_\alpha$.

If F_p is not empty, let β be the largest element of F_p. Let the domain of s_p be contained in some integer m. We must again choose a suitable a_δ to be mod finite contained in every member of $x \cap M$. If we ensure that $a_\delta \cap m = a_\beta \cap m$, then for each $\alpha' \in F_p \cap \beta$, there are $i \in a_{\alpha'}$, $j \in m \cap a_\beta$ and $k = h_{a_{\alpha'}}(i) = h_{a_\beta}(j) < m$ such that $s_p(i) = 0$, $s_p(j) = 1$, and $s_p(k) = 2$. Therefore we must work to ensure that there is an $i \in a_\beta \setminus m$ and distinct $j \in a_\delta \setminus m$ such that $h_{a_\beta}(i) = h_{a_\delta}(j) = k$. Then we can simply extend s_p by sending i to 0, j to 1 and k to 2.

Let $a' = a_\beta \cap m$ and set $\mathcal{S} = \{a \in \mathcal{J} : a \cap m = a'\}$. Observe that $\mathcal{S} \in M_\beta$ where M_β denotes the elementary submodel $M' \in \mathcal{M}_p$ such that $M' \cap \omega_1 = \beta$. Since $a_\beta \in \mathcal{S}$ and a_β is almost contained in every $A \in x \cap M_\beta$, it follows by elementarity, that for every $\mathcal{A} \in [x]^\omega$, there is an $a \in \mathcal{S}$ satisfying $a \prec^* \mathcal{A}$. By Lemma 18, the set $S = \{k : (\forall \mathcal{A} \in [x]^\omega)\ (\exists a \in \mathcal{S})(k \in a)\}$ is a member of $x \cap M_\beta$.

Now define B to be the set of all $k \in S$ for which there is some countable $\mathcal{A} \in [x]^\omega$ and a unique value $g(k)$ such that $h_a(k) = g(k)$ for all $a \in \mathcal{S}$ such that $k \in a$ and $a \prec^* \mathcal{A}$. This set B is also a member of M_β.

If B is a member of x, then by the three set lemma [Wal74, 6.25], there is a partition B_0, B_1, B_2 of B with such that $g[B_i] \cap B_i$ is empty for each i. We may assume that B_0 is a member of x. If B is not a member of x, then let $B_0 = S \setminus B$. Again, we may choose B_0 to be a member of $M_\beta \cap x$.

Therefore a_β is contained mod finite in B_0. Let $k \in a_\beta \cap B_0 \setminus m$ be large enough so that $k \notin g^{-1}(\{0, \ldots, m\})$ and $i = h_{a_\beta}(k) \in a_\beta \setminus m$. If $k \in B$, then there is an $a_\delta = a \in \mathcal{S}$ such that $a \prec^* M$, $k \in a$ and $j = h_a(k) = g(k) \notin B_0$. If $k \notin B$, then $g(k)$ is not defined. Since $k \in S$, it follows that there is an $a_\delta = a \in \mathcal{S}$ such that $a \prec^* M$, $k \in a$, and $h_a(k) = j \neq i$.

It follows that $p' < p$ and $p' \in D_\alpha$ where $F_{p'} = F_p \cup \{\delta\}$, $\mathcal{M}_{p'} = \mathcal{M}_p \cup \{M_\delta\}$, $\mathcal{A}_{p'} = \mathcal{A}_p \cup \{a_\delta\}$ and define $s_{p'} \supset s_p$ with $s_{p'}(i) = 0$, $s_{p'}(j) = 1$ and $s_{p'}(k) = 2$. □

Lemma 20. *The poset P is proper.*

Proof. Let λ be any suitably large cardinal with $P \in H(\lambda)$ and let $\theta, P \in M \prec H(\lambda)$ be a countable elementary submodel. Let $p_0 \in M$, $\delta = M \cap \omega_1$ and $M_\delta = M \cap H(\theta)$. In the proof of Lemma 19, we have shown that there is a $q < p$ such that $M_\delta \in \mathcal{M}_q$.

Let $D \in M$ be a dense subset of P, $q \in D$, $q < p_0$, and $M_\delta \in \mathcal{M}_q$. To show that P is proper, it suffices to show that there is a $q' \in M \cap D$ which is compatible with q.

Let $m_0 \in \omega$ contain the domain of s_q. Let $q_0 = q \cap M$, and $F_q \setminus M = \{\delta_0, \delta_1, \ldots, \delta_{n-1}\}$ listed in increasing order. For $i < n$, let $c_i = a_{\delta_i} \cap m_0$.

Define $T_n \subset {}^n(\omega_1 \times \mathcal{J})$ by $t = \langle (\gamma_i, a_i) \rangle_{i<n} \in T$ if there is a $p \in D$ such that

(1) for each $i < n$, $a_i \cap m_0 = c_i$;
(2) $s_p = s_q$;
(3) F_{q_0} is an initial segment of F_p;
(4) $F_p \setminus F_{q_0} = \{\gamma_i : i < n\}$; and
(5) $t \subset \mathcal{A}_p$.

Observe that $\langle \delta_i, a_{\delta_i} \rangle_{i<n} \in T_n$ and that $T_n \in H(\theta) \cap M = M_{\delta_0}$.

We define by induction on $k < n$, a set T_{n-k}. If T_{n-k} has been defined, then let $\langle(\gamma_i, a_i)\rangle_{i<(n-(k+1))} \in T_{n-(k+1)}$ providing that for each $\mathcal{A} \in [x]^\omega$ and each $\gamma < \omega_1$, there are $\gamma' > \gamma$ and a' such that $a' \prec^* \mathcal{A}$, $\langle(\gamma_i, a_i)\rangle_{i<(n-(k+1))}{}^\frown\{(\gamma', a')\} \in T_{n-k}$. The set $T_{n-k} \in M_\delta$ for each $k < n$.

Since the elements of F_q are separated by the elementary submodels in $\mathcal{M}_q$, it follows that for each $k < n$, $\langle(\gamma_i, a_{\delta_i})\rangle_{i<(n-k)}$ is a member of T_{n-k}. In particular, it follows that $\emptyset \in T_0$. The approach is based on Todorcevic's no S-spaces proof [Tod89, 8.9].

The next step is to inductively find a sequence $\langle(\beta_i, a_i)\rangle_{i<n} \in M_\delta$ so that for each $k < n$, $\langle(\beta_i, a_i)\rangle_{i<n-k}$ is in T_{n-k}. This will identify a condition $p \in M \cap D$ but we will have to have chosen the a_i more carefully in order to arrange that p is compatible with q. This is similar to the proof of Lemma 19.

Let $\mathcal{S}_0 = \{a \in \mathcal{J} : (\exists\gamma)\langle(\gamma, a)\rangle \in T_1\}$. Since a_{δ_0} is in $\mathcal{S}_0 \in M_{\delta_0}$, it follows that $\mathcal{S}_0$ satisfies the hypothesis of Lemma 18. As in the proof of Lemma 19, there is a $k \in a_{\delta_0} \setminus m_0$ such that with $j = h_{a_{\delta_0}}(k)$, there is an $a_0 \in \mathcal{S}_0$ such that $k \in a_0$ and $h_{a_0}(k) = i \neq j$. Fix an element $\langle(\beta_0, a_{\beta_0})\rangle \in T_1 \cap M_\delta$ such that $k \in a_{\beta_0}$ and $h_{a_{\beta_0}}(k) = i$. Let $m_1 \in \omega$ be larger than each of i, j, k.

Let $\mathcal{S}_1 = \{a \in \mathcal{J} : (\exists\beta_1)\ \langle(\beta_0, a_{\beta_0}), (\beta_1, a)\rangle \in T_2\}$. Since $\langle(\beta_0, a_{\beta_0})\rangle \in T_1$, it follows that $\mathcal{S}_1$ satisfies the hypothesis of Lemma 18. We again find $(\beta_1, a_{\beta_1}) \in M_\delta$ such that $\langle(\beta_0, a_{\beta_0}), (\beta_1, a)\rangle \in T_2\}$ and there is a $k \in a_{\delta_1} \cap a_{\beta_1}$ such that $h_{a_{\delta_1}}(k) = j$ is distinct from $h_{a_{\beta_1}}(k) = i$. Let m_2 be greater than each of these values $\{i, j, k\}$.

Proceeding in this way, we can obtain our desired sequence $\langle(\beta_i, a_{\beta_i})\rangle_{i<n} \in T_n \cap M$ together with an increasing sequence $m_0 < m_1 < \cdots < m_{n-1}$ so that for each $\ell < n$, there is a $k_\ell \in a_{\beta_\ell} \cap a_{\delta_\ell} \cap [m_{\ell-1}, m_\ell)$ such that $i_\ell = h_{a_{\beta_\ell}}(k) \neq h_{a_{\delta_\ell}}(k) = j_\ell$ are both in $[m_{\ell-1}, m_\ell)$. Let $p \in D \cap M_\delta$ be chosen to witness that $\langle(\beta_i, a_{\beta_i})\rangle_{i<n} \in T_n \cap M$.

Define a function $s = s_{q'} \supset s_q$ so that for each $\ell < n$, $s(i_\ell) = 0$, $s(j_\ell) = 1$, and $s(k_\ell) = 2$. Let $F_{q'} = F_p \cup F_q$, $\mathcal{M}_{q'} = \mathcal{M}_p \cup \mathcal{M}_q$, and $\mathcal{A}_{q'} = \mathcal{A}_p \cup \mathcal{A}_q$. The properties of s described above ensure that $q' \in P$ and it is clear that $q' < p, q$. □

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