A TUTTE-STYLE PROOF OF BRYLAWSKI'S TENSOR PRODUCT FORMULA

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Dedicated to the memory of Thomas H. Brylawski

Abstract. We provide a new proof of Brylawski's formula for the Tutte polynomial of the tensor product of two matroids. Our proof involves extending Tutte's formula, expressing the Tutte polynomial using a calculus of activities, to all polynomials involved in Brylawski's formula. The approach presented here may be used to show a signed generalization of Brylawski's formula, which may be used to compute the Jones polynomial of some large non-alternating knots. Our proof inspires a generalization of Brylawski's formula to the case when the pointed element is a factor, yielding an unexpected connection to a well-known result on the uniqueness of minimum weight spanning trees in a weighted graph with distinct edge weights.

1. Introduction

As Dominic Welsh noted it during his talk at the Brylawski Memorial Conference [8], very often a statement is shown much more easily, once we know it is true. In this note we wish to prove that this is certainly true for Brylawski's celebrated tensor product formula. We became acquainted with this result while we were trying to develop a way to calculate the Jones polynomial of some large non-alternating knots. Very soon, we were poring over the paper of Jaeger, Vertigan, and Welsh [5], using Brylawski's formula to show that to compute the Jones polynomial of an alternating knot is $\#P$-hard. The knots we wanted to study turned out to be associated to signed graphs that arise via a signed generalization of Brylawski's tensor product operation. To prove a signed generalization of Brylawski's formula [4] for a colored Tutte polynomial that exist by the results of Bollobás and Riordan [1], we developed a Tutte-style "activity calculus" which seems also suitable to provide a new proof of Brylawski's original formula. Alas, we were too late trying to contact Professor Brylawski, and so in this life we will not be able to hear his comments. We can only hope he would like this new approach.

2. Preliminaries

First we review Tutte's original definition of the Tutte polynomial of a connected graph. Replacing the word "spanning tree" with "basis" and using the word "circuit" in the sense "minimal dependent set" yields the obvious generalization to matroids. Throughout this paper we identify graphs and subgraphs with the sets of their edges. Given a connected graph G and a spanning tree T, any edge $e \notin T$ closes a unique cycle in $T \cup \{e\}$. We denote this unique cycle by $C(T, e)$.

Definition 2.1. *Let G be a graph whose edges are bijectively labelled with a totally ordered set of labels, and let T be a spanning tree of G. An* internal *edge $e \in T$ is* internally active *if for any edge $f \neq e$ in G such that $(T \setminus \{e\}) \cup \{f\}$ is a spanning tree of G, the label of e is less than the label of f.*

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Otherwise $e \in T$ *is* internally inactive. *On the other hand, an* external *edge* $f \notin T$ *is* externally active *if* f *has the smallest label among the edges in* $C(T, f)$*. Otherwise,* $f \notin T$ *is* externally inactive.

Note that the first half of the above definition may be restated using the following observation.

Lemma 2.2. *Given a connected graph* G *and a spanning tree* T*, and internal edge* $e \in T$ *may be replaced with an external edge* $f \notin T$ *such that* $(T \setminus \{e\}) \cup \{f\}$ *is also a spanning tree if and only if* $e \in C(T, f)$*.*

We assign weight x to each internally active edge and weight y to each externally active edge, where x and y are independent, commuting variables. The inactive edges have weight 1. Let $w(T)$ be the product of the weights of all edges of G, calculated with respect to the spanning tree T. The Tutte polynomial $T(G)$ is then defined as the sum of all the $w(T)$'s over all spanning trees of G. Tutte's main result [7] is that the Tutte polynomial is independent of the labelling of the edges.

Next we review Brylawski's *tensor product* operation, as defined in [2]. Let M be a matroid and N_e a *pointed* matroid, i.e., a matroid with a distinguished element e. We assume that e is neither a loop nor a coloop, i.e., a *nonfactor* in Brylawski's terminology [2].

Definition 2.3. *The* tensor product $M \otimes N_e$ *is obtained by taking the* 2*-sum of* M *with* N *at each element of* M *with the distinguished element* e *in* N*.*

It is worth observing right away that the 2-sum operation is compatible with duality (see Oxley [6, Proposition 7.1.20]), thus the dual of a tensor product is the tensor product of the respective dual matroids:

$$(M \otimes N_e)^* = M^* \otimes N_e^*. \tag{1}$$

Let us restate this definition in terms of graph theory, such that generalizing as indicated above yields Definition 2.3. A *pointed graph* N_e is a graph N with a distinguished edge e. The *tensor product* $M \otimes N_e$ of a graph M and a pointed graph N_e is then obtained by replacing each edge of M with a copy of $N \setminus e$ in such a way that the endpoints of the distinguished edge e are to be identified with the endpoints of replaced edge of M. Since there is no orientation defined on the edge e, the two different ways of identifying e with the replaced edge in M may lead to tensor products that are not isomorphic as graphs. The tensor product is thus defined only up to the underlying matroid structure. Once we generalize this definition to Definition 2.3, there is of course no ambiguity left.

Remark 2.4. *We use* $N \setminus e$ *to denote the graph (or matroid) obtained by* deleting e *from* N*, whereas* $N \setminus \{e\}$ *stands for the set of edges (elements) of* N *that are different from* e*. As usual,* N/e *is the graph (or matroid) obtained by contracting* e*.*

The main result of our paper is a new proof of Brylawski's following theorem.

Theorem 2.5 (Brylawski)**.** *Let* M *be a matroid and* N_e *a pointed matroid. Then the Tutte polynomial* $T(M \otimes N_e) \in \mathbb{Z}[x, y]$ *may be obtained from the Tutte polynomial* $T(M) \in \mathbb{Z}[x, y]$ *by substituting* $T(N \setminus e)/T_L(N, e)$ *into* x*,* $T(N/e)/T_C(N, e)$ *into* y*, and multiplying the resulting rational expression with* $T_L(N, e)^{r(M)} T_C(N, e)^{|M|-r(M)}$*. That is,*

$$T(M \otimes N_e) = T_L(N, e)^{r(M)} T_C(N, e)^{|M|-r(M)} \cdot T\left(M; \frac{T(N \setminus e)}{T_L(N, e)}, \frac{T(N/e)}{T_C(N, e)}\right).$$

Here $r(M)$ is the rank of M, and the polynomials $T_C(N,e)$ and $T_L(N,e)$ are defined by the system of equations

$$\left.\begin{array}{rcl} T(N/e) - T_C(N,e) &=& (y-1)T_L(N,e) \\ T(N\setminus e) - T_L(N,e) &=& (x-1)T_C(N,e) \end{array}\right\}. \tag{2}$$

The notation $T_C(N,e)$ resp. $T_L(N,e)$ is due to Jaeger, Vertigan and Welsh [5] who used Brylawski's theorem to show that computing the Jones polynomial of even an alternating knot is $\#P$-hard. Note that the system of equations (2) may be rewritten as

$$\left.\begin{array}{rcl} T_C(N,e) + (y-1)T_L(N,e) &=& T(N/e) \\ (x-1)T_C(N,e) + T_L(N,e) &=& T(N\setminus e) \end{array}\right\}$$

which has a unique solution $(T_C(N,e), T_L(N,E))$ by Cramer's rule, since

$$\det\begin{pmatrix} 1 & y-1 \\ x-1 & 1 \end{pmatrix} = 1-(x-1)(y-1) \neq 0.$$

Further study of the polynomials $T_C(N,e)$ and $T_L(N,e)$ may be found in the work of Brylawski and Oxley [3].

3. A Tutte-style definition for $T_C(N,e)$ and $T_L(N,e)$

The main result of this section is that the polynomials $T_C(N,e)$ and $T_L(N,e)$ may be equivalently defined in a way that is completely analogous to Tutte's definition of a Tutte polynomial, in terms of activities. We state and prove our result for connected graphs only, but the extension to all matroids will be obvious and immediate. In the proof of our main result we will use the following facts, summarized as a Lemma.

Lemma 3.1. *Let N be a connected graph, T a spanning tree of N, $e \in T$ an internal edge, e_i and e_j two distinct external edges, such that $C(T,e_i)$ and $C(T,e_j)$ both contain e. Then*

1. *$T_i = (T\setminus\{e\})\cup\{e_i\}$ and $T_j = (T\setminus\{e\})\cup\{e_j\}$ are spanning trees of N as well;*
2. *there exists at least one cycle in $(T\setminus\{e\})\cup\{e_i\}\cup\{e_j\}$;*
3. *any cycle in $(T\setminus\{e\})\cup\{e_i\}\cup\{e_j\}$ must contain both e_i and e_j.*

The set of spanning trees of a connected pointed graph N_e may be partitioned into two disjoint sets depending on whether the special edge e is contained in the spanning tree or not. If e is a nonfactor then the set of spanning trees not containing e is the set of spanning trees of $N\setminus e$ whereas the set of spanning trees containing e is identifiable with the set of spanning trees of N/e via the contraction of e. Given a spanning tree T of N/e, and an edge $f \notin T$, by a slight abuse of terminology we will say that e is contained in $C(T,f)$ if e is contained in the cycle $C(T\cup\{e\},f)$ in N_e. The main result of this section is then the following.

Theorem 3.2. *Let N_e be a pointed graph with e a nonfactor. Label the edges of $N\setminus\{e\}$ with a totally ordered set of labels. Then $T_L(N,e)$ is the polynomial defined by the same rule that defines $T(N\setminus e)$ except that internally active edges on a cycle closed by e will be considered as internally inactive instead. Similarly, $T_C(N,e)$ is the polynomial defined by the same rule that defines $T(N/e)$ except that externally active edges that would close a cycle containing e will be considered as externally inactive instead.*

Proof. It is sufficient to show that the polynomials $T_C(N,e)$ and $T_L(N,e)$, defined by weights based on activities as above satisfy the defining system of equations (2). Furthermore, it is sufficient to show that they satisfy the first equation

$$T(N/e) - T_C(N,e) = (y-1)T_L(N,e) \tag{3}$$

using an argument that relies on matroid properties only. The matroid generalization will then also hold for the dual matroid and the second equation

$$T(N \setminus e) - T_L(N,e) = (x-1)T_C(N,e) \tag{4}$$

is the dual of the first. In fact, under duality not only deletion and contraction exchange role, but also internal (that is, basis) elements become external (that is dual basis) elements, and vice versa.

Let us call a pair (T,g) as *special* if T is a spanning tree of N_e containing e (so it can be identified with a spanning tree of N/e), and $g \notin T$ is an externally active edge such that $e \in C(T,g)$. Equivalently, $g \notin T$ is externally active such that $T \setminus \{e\} \cup \{g\}$ is also a spanning tree.

We shall establish a bijection between the set of special pairs (T,g) and the set of spanning trees of $N_e \setminus e$. Consider a special pair (T,g). The set $T' := T \setminus \{e\} \cup \{g\}$ is a spanning tree of $N_e \setminus e$, and since g is externally active with respect to T, g is the least element in the cycle $C(T,g) = C(T',e)$. Conversely, if T' is a spanning tree of $N \setminus e$, then it is also a spanning tree of N_e (since e is not a coloop) and there exists $g \neq e$ that is the least element in the cycle $C(T',e)$ (Note that e, as the distinguished edge, does not have a label since it is not actually present in N/e or in $N \setminus e$). Then $T := T' \setminus \{g\} \cup \{e\}$ is a spanning tree of N_e containing e, and g is externally active with respect to T, hence (T,g) is a special pair.

To compute $T(N/e) - T_C(N,e)$ we sum over all spanning trees of N_e that contain e. If T is a fixed spanning tree, the weight of each edge g is the same when computing $T(N/e)$ and $T_C(N,e)$, except when (T,g) is a special pair. Thus we obtain a nonzero contribution if and only if there is at least one $g \notin T$ such that (T,g) is a special pair. The right hand side of equation (3) is a sum over all spanning trees T' of $N \setminus e$. Thus it is sufficient to show the following: if T is a spanning tree of N_e containing e, and $g_1,\ldots,g_k$ are all of the edges such that (T,g_i) form a special pair, then the contribution of T to the left hand side is the same as the total contribution of all trees $T_i := T \setminus \{e\} \cup \{g_i\}$ to the right hand side. Without loss of generality we may assume that $g_1,\ldots,g_k$ are listed in increasing order of labels.

Since $g_1,\ldots,g_k$ are all externally active with respect to N/e but are all considered inactive when computing $T_C(N,e)$, the contribution of T to the left hand side of equation (3) is $y^k - 1$ times the contribution of the edges that are not in the set $\{g_1,\ldots,g_k\}$. Since $y^k - 1 = (y-1)(1+y+y^2+\cdots+y^{k-1})$, it is sufficient to prove the following two statements:

1. The product of the weights of the edges $\{g_1,\ldots,g_k\}$ in the contribution of any T_i to $T_L(N,e)$ is y^{i-1}.
2. The weight of any edge $f \notin \{e, g_1,\ldots,g_k\}$ is the same in the contribution of T to $T(N/e)$ as in the contribution of any T_i to $T_L(N,e)$.

Regarding the first statement, since g_i is internal to T_i and is in $C(T_i,e)$, it is considered inactive when computing $T_L(N,e)$ (by the exception rule) and thus contributes a factor of 1. For $j > i$, the cycle $C(T_i,g_j)$ contains both g_i and g_j, and g_i has the smallest label in this cycle by our assumption of the ordering of the edges. So g_j is externally inactive, contributing a factor of 1. For $j < i$ the cycle $C(T_i,g_j)$ again contains both g_i and g_j, but in this case g_j has the smallest label in the cycle, hence it is externally active. The exception rule does not apply since e is not in $C(T_i, g_j)$, so g_j contributes a factor of y. Thus the total contribution of T_i to $T_L(N, e)$ is y^{i-1} times the contribution of the edges not in $\{g_1, \dots, g_k\}$.

To prove the second statement, we consider the following three cases, keeping in mind that for $f \notin \{e, g_1, ..., g_k\}$, f is either simultaneously internal to both T and T_i, or simultaneously external to both T and T_i.

Case 1. $f \in C(T, g_i) = C(T_i, e)$. In this case f is internally inactive in T since (T, g_i) is a special pair (so the label of g_i is less than that of f). f is also internally inactive in the contribution of T_i to $T_L(N, e)$, for even if it would normally be active in T_i, it would be considered inactive according to the exception rule for $T_L(N, e)$.

Case 2. $f \in T \setminus C(T, g_i)$. In this case f is internal and is compared with the same edges in both T and T_i. If $f \in C(T, g)$ for some external edge g, then $g \neq g_i$ (since $f \notin C(T, g_i)$), so g is also external to T_i. The exception rule does not apply to f in the contribution to $T_L(N/e)$ since $f \notin C(T_i, e)$. Conversely, if $f \in C(T_i, g)$ for some $g \notin T_i$, then $g \neq e$ (since $f \notin C(T_i, e)$), so g is also external to T. Since f is compared to the same edges in both T and T_i, it is either simultaneously active or simultaneously inactive with respect to both T and T_i, thus its contributions to $T(N/e)$ and $T_L(N, e)$ are the same.

Case 3. $f \notin T \cup T_i$. If $C(T, f) = C(T_i, f)$ then clearly f is either simultaneously active or simultaneously inactive with respect to both T and T_i. If $C(T, f) \neq C(T_i, f)$ then $e \in C(T, f)$ and $g_i \in C(T_i, f)$. Since $e \in C(T, f)$ and the pair (T, f) is not special, f must be externally inactive with respect to T. And since $g_i \in C(T_i, f)$ and (T, g_i) is special, it follows that the label of g_i is less than the label of f, hence f is externally inactive with respect to T_i. □

One of the most important consequences of Theorem 3.2 is the following.

Corollary 3.3. *The definition of the polynomials $T_C(N, e)$ and $T_L(N, e)$ by weights based on activities is* independent of the labelling.

4. A Tutte-style proof of Brylawski's theorem

Using the Tutte-style definition of $T_C(N, e)$ and $T_L(N, e)$ given in Theorem 3.2 we are able to provide a Tutte-style proof of Theorem 2.5. As in the rest of the paper, we rephrase and prove the statement for connected graphs only, the generalization to matroids is obvious and immediate.

Theorem 4.1. *Let M be a connected graph and N_e a pointed graph with e a nonfactor. Then the Tutte polynomial $T(M \otimes N_e) \in \mathbb{Z}[x, y]$ is given by*

$$T(M \otimes N_e) = T_L(N, e)^{r(M)} T_C(N, e)^{|M| - r(M)} \cdot T\left(M; \frac{T(N \setminus e)}{T_L(N, e)}, \frac{T(N/e)}{T_C(N, e)}\right).$$

Here $r(M)$ is the number of edges in any spanning tree of M.

Proof. Assume M has m edges, bijectively labelled with the set of labels $\{1, 2, \dots, m\}$, ordered by the usual order. By Tutte's result [7], the polynomial $T(M; T(N \setminus e)/T_L(N, e), T(N/e)/T_C(N, e))$ may be computed as the total weight $w'(T)$ of all spanning trees T of M, where $w'(T)$ is the product of the weight of all edges of M such that each internally active edge of M contributes a factor of

$T(N \setminus e)/T_L(N,e)$ and each externally active edge of M contributes a factor of $T(N/e)/T_C(N,e)$. Inactive edges contribute a factor of 1.

The number of internal edges is $r(M)$, the number of external edges is $|M|-r(M)$. Thus, multiplying $T_L(N,e)^{r(M)}T_C(N,e)^{|M|-r(M)}$ into $T\left(M; T(N \setminus e)/T_L(N,e), T(N/e)/T_C(N,e)\right)$ yields a polynomial that may be computed as the total weight $w''(T)$ of all spanning trees T of M, where $w''(T)$ is the product of the weight of all edges of M such that

- each internally active edge of M contributes a factor of $T(N \setminus e)$;
- each internally inactive edge contributes a factor of $T_L(N,e)$;
- each externally active edge contributes a factor of $T(N/e)$; and
- each externally inactive edge contributes a factor of $T_C(N,e)$ to $w''(T)$.

It is sufficient to show that the sum of all weights $w''(T)$ also equals $T(M \otimes N_e)$. For that purpose we first fix an appropriate labeling on $M \otimes N_e$. Assume $N \setminus e$ has n edges, bijectively labelled with $\{1, 2, \ldots, n\}$, ordered by the usual order. Let us denote by e_i the edge of M with label i and let us denote by N_i the copy of $N \setminus e$ that replaces e_i in $M \otimes N_e$. The graph $M \otimes N_e$ (identified with its set of edges) is then the disjoint union of all N_i's. Label an edge of $M \otimes N_e$ with $i + j/(n+1)$ if this edge belongs to N_i and it is identified with the edge labelled j in $N \setminus e$. Under this labelling, edges belonging to the same N_i form a set of consecutively labelled edges.

Consider a spanning tree T' of $M \otimes N_e$. This tree induces a spanning tree T of M in a natural way: for a nonloop edge we set $e_i \in T$ if and only if T' contains a path in N_i that connects the two vertices of the edge e_i in M replaced by N_i. Conversely, a spanning tree T of M can be extended to a spanning tree T' of $M \otimes N_e$ by replacing each $e_i \in T$ by a spanning tree of $N \setminus e$ (these contain a path connecting the endpoints of e_i) and each $e_i \notin T$ by a spanning tree of N/e (these do not contain a path connecting the endpoints of e_i if e_i is not a loop). Any spanning tree T' of $M \otimes N_e$ obtained from the spanning tree T of M is called an *offspring* of T and the tree T is called a *parent tree* of T'. We now consider the total contribution of all the offsprings of a parent tree T in M to $T(M \otimes N_e)$. It is sufficient to show that, when we compute this contribution, the total weight of all edges contained in the same N_i is the same as the contribution of e_i to $w''(T)$.

Case 1 $e_i \in T$ is internally active. Then T' contains a spanning tree T_i of N_i. Any external edge $f \in N_i \setminus T'$ closes a cycle with T_i, thus the question whether f is externally active is decided "locally", within N_i. On the other hand, if $f \in N_i$ is internally inactive then there exists an edge g whose label is smaller than the label of f such that $f \in C(T', g)$. We must have $g \in N_i$, otherwise $e_i \in C(F, g)$ and $e_i \in F$ is internally inactive. Therefore, the total contribution of N_i to $T(M \otimes N_e)$ is $T(N \setminus e)$.

Case 2 $e_i \in T$ is internally inactive. Similarly to the previous case, the question whether an external edge of N_i is active is decided within N_i. Since e_i is internally inactive, there is an $e_j \notin T$ satisfying $j < i$ such that $C(T, e_j) \subseteq M$ contains e_i. Consequently, for any offspring T' of T, there is a $g \in N_j$ whose label is less than i such that g closes a cycle $C(T', g)$ in $M \otimes N_e$ and $N_i \cap C(T', g) \neq \emptyset$. Any edge of $N_i \cap T' \cap C(T', g)$ is thus internally inactive. Note that the edges in $N_i \cap T' \cap C(T', g)$ are precisely those in the set $C(T', e_i) \subseteq N_i$ and the exception rule of $T_L(N, e)$ apply to exactly these edges. It is easy to check that the activity of the remaining internal edges can be decided locally. Hence, for each internally inactive $e_i \in T$, N_i contributes a factor of $T_L(N, e)$.

Case 3 $e_i \notin T$ is externally active. Similarly to Case 1, internal and external activities are decided locally within N_i. Hence N_i contributes a factor of $T(N/e)$.

Case 4 $e_i \notin T$ is externally inactive. Similarly to Case 2, we may show that N_i contributes a factor of $T_C(N,e)$. □

5. Extending Brylawski's theorem to the case when e is a factor

Brylawski [2] left the tensor product $M \otimes N_e$ undefined in the case when e is a factor. There is nothing that prevents the extension of the operation to this degenerate case. In fact, we have the following simple description.

Lemma 5.1. *If e is a factor of e then $M \otimes N_e$ is isomorphic to the direct sum of $|M|$ copies of $N \setminus e = N/e$.*

Proof. We prove the statement for graphs only (the generalization to matroids is immediate), and only for the case when e is a coloop. The case when e is a loop is the dual of the previous and follows from (1). Without loss of generality we may assume that e is in its own connected component (if not, splitting a cut vertex in a graph does not change the underlying matroid structure). The graph $M \otimes N_e$ is then a graph that consists of $|M|$ disjoint copies of $N \setminus e = N/e$. □

Corollary 5.2. *If e is a factor then*

$$T(M \otimes N_e) = T(N \setminus e)^{|M|} = T(N/e)^{|M|}.$$

Next we extend the definition of $T_C(N,e)$ and $T_L(N,e)$ to the case when e is a factor.

Definition 5.3. *We set $T_L(N,e) := T(N \setminus e)$ and $T_C(N,e) := 0$ when e is a loop, and we set $T_L(N,e) := 0$ and $T_C(N,e) := T(N/e)$ when e is a coloop.*

In a way, Definition 5.3 may be considered a natural extension of the Tutte-style definition given in Theorem 3.2. For example, if e is a loop then there is no other edge on a cycle closed by e, so $T_L(N,e) = T(N \setminus e)$. The "explanation" of $T_L(N,e) = 0$ is a little trickier. What we need to keep in mind that the factors $T_C(N,e)$ appear as contributions of spanning trees *containing* e, and no spanning tree contains a loop. This heuristic, argument may not seem completely convincing, however, using this definition, we may extend Brylawski's formula as follows.

Theorem 5.4. *Let M be any matroid and N_e be any pointed matroid. Then the Tutte polynomial $T(M \otimes N_e)$ may be obtained by substituting $u = T_L(N,e)$ and $v = T_C(N,e)$ into the polynomial*

$$u^{r(M)} v^{|M|-r(M)} \cdot T\left(M; \frac{T(N \setminus e)}{u}, \frac{T(N/e)}{v}\right).$$

Proof. We will assume that M is a graph and N_e is a pointed graph, the generalization to all matroids is obvious. Observe first that the expression $u^{r(M)} v^{|M|-r(M)} \cdot T\left(M; T(N \setminus e)/u, T(N/e)/v\right)$ is in deed a polynomial of u and v. In fact, as observed in the proof of Theorem 4.1, we may use Tutte's formula to compute $T\left(M; T(N \setminus e)/u, T(N/e)/v\right)$ as a total weight of spanning trees of M, and the number of internal edges will be $r(M)$. Thus the highest power of the rational expression $T(N \setminus e)/u$ appearing in a term is $r(M)$, multiplying with $u^r(M)$ yields a non-negative power of u. Similarly, the number of external edges being $|M| - r(M)$ guarantees that multiplying by $v^{|M|-r(m)}$ eliminates all occurrences of v in the denominator.

We only need to prove our generalization for the case when e is a factor, and without loss of generality we may assume that e is a coloop. (The other case follows by duality.) In analogy to our proof of Theorem 4.1, when we substitute $u = T_L(N,e)$ and $v = T_C(N,e)$ into $u^{r(M)}v^{|M|-r(M)} \cdot T(M; T(N \setminus e)/u, T(N/e)/v)$ we obtain the total weight $w'''(T)$ of all spanning trees T of M where

- each internally active edge of M contributes a factor of $T(N \setminus e)$;
- each internally inactive edge contributes a factor of $T_L(N,e) = 0$;
- each externally active edge contributes a factor of $T(N/e) = T(N \setminus e)$; and
- each externally inactive edge contributes a factor of $T_C(N,e) = T(N/e) = T(N \setminus e)$ to $w'''(T)$.

In particular $w'''(T) = 0$ unless all internal edges of T are active. Each T having only internally active edges contributes a factor of $T(N \setminus e)^{|M|} = T(N/e)^{|M|}$. Therefore we only need to show that there is a unique spanning tree T having only internally active edges. Without loss of generality we may assume that the edges of M are bijectively labeled with the set if labels $\{1, 2, \ldots, n\}$, ordered by the natural order. Consider the label i as the "cost" of the edge. It is easy to see that T has only internally active edges, if and only if it is a minimum cost spanning tree. Thus our theorem follows from the well-known fact that a connected graph whose edges are labeled with distinct costs has a unique minimum weight spanning tree. (See, e.g. West [8, Exercise 2.3.7], the suggested proof easily generalizes to matroids.) □

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